THE APPROXIMATION OF NUMBERS AS SUMS OF RECIPROCALS

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1. Introduction. Every positive real number $x<1$ admits a unique expansion of the form

$$x = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots, \tag{1}$$

where a_1, a_2, a_3, and so forth, are integers so chosen that after i terms, when the sum x_i has been obtained, a_{i+1} is the least integer such that $x_i + 1/a_{i+1}$ does not exceed x. This expansion will be denoted as the *R-expansion of the number* x. The approximation as far as x_i, is obviously closer to x than $1/(a_i^2 - a_i)$, since

$$x - x_i < \frac{1}{a_i - 1} - \frac{1}{a_i}$$

by the choice of a_i. Thus it is apparent that in every R-expansion, $a_{i+1} = a_i^2 - a_i + \epsilon_i$, where ϵ_i is a positive integer.

Every expansion of the form (1), where $a_{i+1} = a_i^2 - a_i + \epsilon_i$, $\epsilon_i \geqq 1$ (apart from the question as to whether it is the R-expansion of some number), converges. In particular, the R-expansion of x converges to x. As an indication of the rapidity of convergence, note that every such expansion is dominated by the expansion beginning with $a_1 = 2$ and $\epsilon_i = 1$ for $i \geqq 1$, that is,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots.$$

This latter series can be shown to converge to 1.

Proof: Let $1/a_i$ be a particular addend. If it has the property that $1/(a_i - 1)$ in its place would have made x_i equal to 1, then the next addend will also have that property for x_{i+1}. The reason is that the next addend is $1/(a_i^2 - a_i + 1)$ and if it were $1/(a_i^2 - a_i)$, then

$$\frac{1}{a_i} + \frac{1}{a_i^2 - a_i} = \frac{1}{a_i - 1},$$

and thus by the hypothesis of the induction, the next addend will have that property, and hence all the addends, because it is true for $1/a_1$ and $1/a_2$. But by the property of the addends which was just established,

$$1 - x_i = \frac{1}{a_i - 1} - \frac{1}{a_i} = \frac{1}{a_i^2 + a_i},$$

which is less than any preassigned η for all $i >$ some i_0; so $x_i \to 1$.

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* Mathematical Tables Project, National Bureau of Standards.

2. The principal theorems.

THEOREM I. *Every infinite expansion of the form* (1) *where* $a_{i+1}=a_i^2-a_i+\epsilon_i$, *where* $\epsilon_i>1$ *for infinitely many values of* i (*this condition on the* ϵ_i *is also necessary*), *is actually the R-expansion of the number* x *to which it converges.**

Proof: If the theorem were not true, compare the above mentioned expansion of x, say (R'), with the R-expansion of x, say (R), and let $1/a_i$ be the first term of (R') that differs from the corresponding term of (R). By the definition of the R-expansion for x, $1/a_i$ must be less than that corresponding term, say $1/(a_i-b)$. But $x-x_{i-1}$, the remainder after $i-1$ terms in the R-expansion, satisfies

$$(2) \qquad x - x_{i-1} \geqq \frac{1}{a_i - b} \geqq \frac{1}{a_i - 1} > \frac{1}{a_i} + \frac{1}{a_i^2 - a_i + \epsilon_i} + \cdots,$$

this last inequality being obtained by applying the inductive reasoning of the preceding paragraph to the condition that there is always an $\epsilon_j>1$ for $j>i_0$ due to the infinitude of $\epsilon_i>1$ in (R'). But this is a contradiction, since

$$x - x_{i-1} = \frac{1}{a_i} + \frac{1}{a_i^2 - a_i + \epsilon_i} + \cdots.$$

If there is an expansion of the type (1) with $\epsilon_i=1$ for all $i\geqq i_0$, it is not the R-expansion of any number; but it will agree with the R-expansion of the number to which it converges, up to the term before $1/a_{i_0}$. If the expansion did not agree, it would disagree at a place j where there is a subsequent $\epsilon_i>1$, and again there would arise the same contradiction as before, since, in its remainder expression (the extreme right member of (2) with i replaced by j), the presence of a single $\epsilon_i>1$ makes it less than $1/(a_j-1)$ and hence less than $x-x_{j-1}$. Then $1/a_{i_0}$ plus all the succeeding terms will equal $1/(a_{i_0}-1)$ which will also be the last addend in the terminating R-expansion.

For any rational number a/b there is the very natural question as to whether its R-expansion might continue indefinitely, just as

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots.$$

That such is not the case is contained in the fundamental theorem which follows.

THEOREM II. *The R-expansion of any rational number must terminate.*

Proof: Consider a/b beginning with $1/a_1+\cdots$, so that

$$\frac{a}{b} - \frac{1}{a_1} = \frac{aa_1 - b}{ba_1}.$$

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* For a finite expansion of the form (1) to be the R-expansion, the condition $a_{i+1}=a_i^2-a_i+\epsilon_i$, $\epsilon_i\geqq 1$, is necessary and sufficient.

From the choice of a_1,

$$\frac{a}{b} < \frac{1}{a_1 - 1}, \qquad aa_1 - a < b, \qquad 0 < aa_1 - b < a,$$

so that, disregarding cancellation which only helps the argument, the first remainder a'/b' has $a' < a$. Repetition of this reasoning on a'/b' leads to the next remainder a''/b'' where $a'' < a'$, and the decreasing sequence of positive numerators in the succession of remainders must end in a 1. (If the last remainder is assumed to be p/q in lowest terms with $1 < p < q$, one could continue with $1/([q/p]+1)+\cdots$, thereby contradicting the fact that p/q is the last remainder.) Hence every rational number has a terminating R-expansion and every non-terminating R-expansion must represent an irrational number.

As an immediate corollary to Theorems I and II, a convergent infinite series such as

$$\frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^8} + \cdots + \frac{1}{3^{2^n}} + \cdots$$

cannot be equal to a rational number. In fact, no infinite series of the type $\sum_{n=1}^{n=\infty} a_n^{-b_n}$, where a_n and b_n are any non-decreasing sequences of integers, each greater than or equal to 2, can equal a rational number.

3. Alternative methods. Besides the R-expansions, there are similar alternative methods of expanding any x, $0 < x < 1$, as a sum of reciprocals. Thus to x_i, instead of $1/a_{i+1}$, one might add either $1/a_{i+1}$ or $1/(a_{i+1}-1)$, depending upon which gives a smaller value to $|x - x_{i+1}|$. Such an expansion, denoted as the $\overline{R}$-expansion, would involve an irregular sequence of signs in its terms, since $1/a_j$ would be subtracted from x_{j-1} whenever $x_{j-1} > x$. The absolute value of the remainder in stopping at $1/a_j$ is not greater than $1/2a_j(a_j-1)$. If $\pm 1/a_i$ is a particular term in the $\overline{R}$-expansion, the next term is always $\pm 1/(2a_i^2 - 2a_i + \epsilon_i)$, where $\epsilon_i \geqq 0$. This shows that the $\overline{R}$-expansion is even more rapidly convergent than the R-expansion. The $\overline{R}$-expansion is unique except for certain rational numbers where the last two terms might be either

$$\pm\frac{1}{a_i} \pm \frac{1}{2a_i(a_i-1)} \quad \text{or} \quad \pm\frac{1}{a_i - 1} \mp \frac{1}{2a_i(a_i-1)}.$$

Further comparison of the $\overline{R}$-expansion with the R-expansion reveals another important difference. Unlike Theorem I for the R-expansion, every expansion where $\pm 1/a_i$ is followed by $\pm 1/\{2a_i(a_i-1)+\epsilon_i\}$, $\epsilon_i \geqq 0$, is not necessarily an $\overline{R}$-expansion ("an" instead of "the" to cover the cases where there is no unique $\overline{R}$-expansion) of the number to which it converges. Among the many trivial illustrations of this point are the following: I. $a_1 = 1$, all $\epsilon_i = 1$ (not even convergent); II. $a_1 = 1$, $\epsilon_1 = 2$, $\epsilon_2 = 0$; III. The first two terms may be $1/2+1/4+\cdots$, whereas the $\overline{R}$-expansion might begin with 1. For a non-trivial instance consider 111/264 when it is expressed as $1/3+1/12+1/264$,

where obviously $1/a_i$ is followed by $1/\{2a_i(a_i-1)+\epsilon_i\}$, $\epsilon_i \geqq 0$, where $a_1=3$, $\epsilon_1=\epsilon_2=0$. But the $\overline{R}$-expansion of $111/264$ is $1/2-1/13-1/381+\cdots$.

The most slowly convergent R-expansion is

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1807}+\cdots,$$

where the first five terms give an approximation that differs from the true value by more than 3×10^{-7}. The most slowly convergent $\overline{R}$-expansion is

$$\frac{1}{2}\pm\frac{1}{4}\pm\frac{1}{24}\pm\frac{1}{1104}\pm\frac{1}{2435424}\pm\cdots,$$

where the first five terms come nearer to the true value than 1×10^{-13}.* Another good illustration for the purpose of comparison is afforded by the first four terms in the R- and $\overline{R}$-expansions for the decimal part of π, where the advantage of the $\overline{R}$-expansion is evident: Thus

$$(R)\quad \pi = 3+\frac{1}{8}+\frac{1}{61}+\frac{1}{5020}+\frac{1}{1285\ 41455}+\cdots,$$

with an error greater than 6×10^{-18}, while

$$(\overline{R})\quad \pi = 3+\frac{1}{7}-\frac{1}{791}-\frac{1}{3748629}+\frac{1}{15164\ 89608\ 87729}-\cdots,$$

with an error less than 1×10^{-30}.

There is a theorem for the $\overline{R}$-expansion that is similar to Theorem II. It is:

THEOREM III. *Every rational number has a terminating $\overline{R}$-expansion.*

Proof:

$$\frac{a}{b}-\frac{1}{a_1}=\frac{a_1a-b}{ba_1}.$$

If

$$\frac{a}{b}>\frac{1}{a_1},\quad \text{then}\quad \frac{a}{b}<\frac{1}{a_1-1}$$

as in Theorem II, and the next remainder a'/b' has $a'<a$. However, if $a/b<1/a_1$, since $1/(a_1+1)<a/b$ (otherwise $1/(a_1+1)$ would be closer to a/b than $1/a_1$), it follows that $b<aa_1+a$, and $aa_1-b>-a$. In view of the fact that $aa_1-b<0$, $|a'|<a$.

Still another efficient method of expanding in reciprocals, which leads to an alternating series, is to write

* It should be noted that the expansion, $1-1/4\pm$ same terms as above, converges equally "slowly."

$$x = \frac{1}{\left[\frac{1}{x}\right]} - r_1, \qquad r_1 = \frac{1}{\left[\frac{1}{r_1}\right]} - r_2, \text{ and so on.}$$

The accuracy of the ith approximation is similar to that of the R-expansions, namely, $1/(a_i^2 - a_i)$. Again, every rational number a/b is seen to have a finite expansion, since

$$\frac{a}{b} - \frac{1}{a_1} = \frac{aa_1 - b}{ba_1} < 0,$$

and from the fact that

$$\frac{1}{a_1} > \frac{a}{b} > \frac{1}{a_1 + 1},$$

we obtain $-a < aa_1 - b < 0$, so that $|a'| < a$.

In the R-expansion of a number, stopping at $1/a_i$ leads to an error of approximately $1/a_i^2$ for a_i only moderately large. When those first i terms are combined into a single fraction, the unreduced denominator is seen to be $k_i a_i^2$, where k_i is considerably less than 1. In every case that x_i is written as p/q, the closeness to x is less than $1/q$, which makes the successive approximations of an R-expansion similar to decimal approximations in the sense that the error in a decimal approximation is usually about a half-unit in the last place. Thus, the R-expansion cannot give rise to what occurs in the convergent series

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

where a fraction of enormous size will yield only 1% accuracy. An approximation p/q obtained from an R-expansion is not as uniformly good as p'/q', obtained from a simple continued fraction where the closeness of approximation is $\sim 1/q'^2$. But the R-expansion converges much more rapidly, and it is often unnecessary to combine the separate reciprocals into a fraction with a denominator larger than the last a_i. Similar remarks hold for the $\overline{R}$-expansions, but even more favorably.

4. Applications. A useful application of the R and $\overline{R}$-expansions is in solving for the positive roots of algebraic (and also transcendental) equations, by a well-known variant of the Newton-Raphson method. By using the R- and $\overline{R}$-expansions less digital work is involved, and the result is exhibited in neat form as a sum of a few reciprocals of integers. It is of interest to compare the use of the R- and $\overline{R}$-expansions with a similar process described by T. A. Pierce in his paper, *On an Algorithm and its Use in Approximating Roots of Algebraic Equations*, this MONTHLY (36) 1929, pp. 523–525. Pierce discusses the expansion

$$x = \frac{1}{p_1}(1 - \frac{1}{p_2}(1 - \frac{1}{p_3}(1 - \cdots$$

where at each stage p_i is determined as the largest integer that will permit the succeeding factor in parentheses to be less than 1. His algorithm has the advantage of involving smaller numbers than the use of the R- or $\overline{R}$-expansion, but when written as a sum of reciprocals it obviously cannot converge as rapidly as either the R- or $\overline{R}$-expansion, due to the fact that each denominator is restricted to being a multiple of the preceding.

For purpose of comparison, consider the same equation that is in Pierce's paper, namely,

$$x^3 - 5x + 2 = 0.*$$

(For reference, its real root between 0 and 1, to 38 decimals, is .41421 35623 73095 04880 16887 24209 69807 857.) Pierce's method of approximation gives successively

$$x \equiv x_1 = \tfrac{1}{2}(1 - x_2),\dagger \qquad -x_2^3 + 3x_2^2 + 17x_2 - 3 = 0,$$

from which

$$x_2 = \tfrac{1}{5}(1 - x_3), \qquad x_3^3 + 12x_3^2 - 452x_3 + 64 = 0,$$

from which

$$x_3 = \tfrac{1}{7}(1 - x_4), \qquad -x_4^3 + 87x_4^2 + 21977x_4 - 111 = 0,$$

from which

$$x_4 = \frac{1}{197}(1 - x_5).$$

The first four terms obtained by Pierce's method, that is,

$$\frac{1}{2}\left(1 - \frac{1}{5}\left(1 - \frac{1}{7}\left(1 - \frac{1}{197}\right)\right)\right),$$

give an approximation that is correct only to within 4 units in the 7th decimal place. Pierce erred in giving .41421 *3564* instead of .41421 *3198* as the fourth approximation, and thus incorrectly claimed that the fourth approximation is correct to within 2 units in the 9th decimal place. A further application of Pierce's process leads to

$$x_5^3 + 17136x_5^2 - 8529\ 39668x_5 + 4286128 = 0, \qquad x_5 = \frac{1}{199}(1 - x_6),$$

and the fifth approximation, .41421 35623 73142, due to the exceptionally small value of x_6, just happens to agree with the true value of the root of 13 decimal places.

Application of the R-expansion method leads to

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* Notice that this equation is different from that well-known and "overworked" test equation $x^3-2x-5=0$, whose real root has been calculated to 152 decimals.

† Here x_i is used differently than in the text preceding this example.

$$x \equiv x_1 = \tfrac{1}{3} + x_2$$

since x is between $1/2$ and $1/3$; then

$$27x_2^3 + 27x_2^2 - 126x_2 + 10 = 0$$

from which

$$x_2 = \frac{1}{13} + x_3, \qquad 59319x_3^3 + 73008x_3^2 - 266643x_3 + 1054 = 0$$

from which

$$x_3 = \frac{1}{253} + x_4, \qquad 9606283\ 17363x_4^3 + 119\ 37026\ 24829x_4^2$$
$$- 430\ 86992\ 40846x_4 + 197\ 46514 = 0$$

from which

$$x_4 = \frac{1}{218201} + x_5.$$

The fourth approximation here, namely,

$$\frac{1}{3} + \frac{1}{13} + \frac{1}{253} + \frac{1}{218201},$$

is accurate to within 2 units in the 11th decimal place (compare with Pierce above). Another step gives

$$x_5 = \frac{1}{6\ 13235\ 43802} + x_6,$$

which leads to a fifth approximation that is correct to within 2 units in the 22nd decimal place (compare again with Pierce above). Work can be saved in obtaining the last desired term (also in Pierce's method) by noting that it is not necessary to find more than the linear part of the last equation, and even there it is not necessary to retain more than a certain number of significant figures in the calculation. Thus if, in the present example, one did not wish to go beyond x_4, instead of the above cubic equation in x_4, with such large coefficients, it would have sufficed to know only

$$\cdots - 430\ 870 \times 10^7 x_4 + 197\ 465 \times 10^2 = 0.$$

In entirely similar fashion, application of the $\overline{R}$-expansion would yield a fourth approximation of

$$\frac{1}{3} + \frac{1}{12} - \frac{1}{408} - \frac{1}{450\ 832},$$

which differs from the root by less than 2 units in the 12th decimal place (com-

pare with both Pierce and the R-expansion), and the fifth approximation which would be obtained from the fourth by subtracting 1/62 70135 66048, would be accurate to within 1 unit in the 24th decimal place (compare again with both Pierce and R-expansion).

Either the R- or $\overline{R}$-expansion can be employed to find a complex root of an equation, merely by applying it to the root's real and imaginary part, each considered separately.

GEODESIC PERSPECTIVITIES UPON A SPHERE*

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1. Conformal perspectivities. Recently Kasner and the author developed a proof of the following proposition in the real and imaginary domains:

If a surface admits a conformal perspectivity upon a given sphere from a given point, then the surface is also a sphere (or plane) homothetic to the original with respect to the point of perspectivity **[1]**.

That is, the point of perspectivity and the two centers of the spheres are collinear, and the distances of the two centers from the point of perspectivity are proportional to the two radii. Moreover, the spheres are inscribed in the tangent cone (real or imaginary) of the original sphere with vertex at the point of perspectivity.

It is possible to deduce as a corollary from the above theorem the converse of Ptolemy's theorem on stereographic projection which states that the only conformal perspectivities upon a plane are Ptolemy's stereographic projection, and the obvious limiting case of a parallel plane **[2]**.

2. Gnomic projection. The following result has also been proved previously:

If more than $3\infty^1$ *geodesics of a surface are projected into straight lines on a plane under a perspectivity, then all geodesics project into straight lines and the surface is a sphere (and the obvious limiting case of any plane, parallel or not), with center at the point of perspectivity* **[3]**.

Surfaces may be classified into the following four distinct classes according to the number of geodesics which are projected into straight lines on a plane by a perspectivity:

1. The non-ruled surfaces. At most ∞^1.
2. The ruled surfaces excluding the quadrics. There are always ∞^1 (the rulings) and at most $2\infty^1$.
3. The quadrics excluding the gnomic projections of spheres. There are always $2\infty^1$ (the two systems of rulings) and at most $3\infty^1$.
4. The gnomic projections of spheres; and planes. All ∞^2.

* Presented to the American Mathematical Society, April, 1946.

3. The fundamental theorem. In the present article, we shall find all surfaces for which there exist perspectivities upon a sphere such that all the geodesics correspond to the great circles. We shall prove the following result.

FUNDAMENTAL THEOREM. *If a surface admits a perspectivity upon a sphere from a given point such that all the geodesics of the surface correspond to the great circles of the sphere, then the surface is a sphere (or plane) homothetic to the original with respect to the point of perspectivity.*

This includes the characterization of gnomic projection which is stated above, as a special case.

4. Part I of the proof. Let (x, y, z) denote cartesian coördinates of a point in space. Take the center of perspectivity as the origin (0, 0, 0). No loss in generality is suffered by assuming the given sphere S to be of unit radius and its center on the z-axis, for any other surface which is a geodesic perspectivity of S is one also of any other sphere homothetic to S with respect to the origin. Accordingly the equation of our sphere S may be written as

$$X^2 + Y^2 + (Z - c)^2 = 1. \tag{1}$$

For the surface Σ, we shall use cylindrical coordinates (ρ, θ, z), where $x=\rho \cos \theta$, $y=\rho \sin \theta$. Then $z=f(\rho, \theta)$; and $p=\partial z/\partial\theta$, $q=\partial z/\partial\rho$, $r=\partial^2 z/\partial\theta^2$, $s=\partial^2 z/\partial\theta\partial\rho$, $t=\partial^2 z/\partial\rho^2$.

The perspective image (X, Y, Z) on the sphere S of a point (ρ, θ, z) on the surface $\Sigma: z=f(\rho, \theta)$, is

$$X = \frac{\rho \cos\theta}{\lambda}, \qquad Y = \frac{\rho \sin\theta}{\lambda}, \qquad Z = \frac{z}{\lambda}, \tag{2}$$

where λ is a function of (ρ, θ) defined by the quadratic equation

$$a\lambda^2 - 2cz\lambda + (\rho^2 + z^2) = 0; \qquad a = c^2 - 1. \tag{3}$$

For our further work, it is advantageous to find the total derivatives of λ with respect to θ. The first order total derivative λ' is given by

$$(a\lambda - cz)\lambda' + (z - c\lambda)(p + q\rho') + \rho\rho' = 0. \tag{4}$$

The second order total derivative λ'' is defined by

$$\begin{aligned}(a\lambda - cz)\lambda'' + a\lambda'^2 - 2c(p + q\rho')\lambda' + (p + q\rho')^2 + \rho'^2 \\ + (z - c\lambda)(r + 2s\rho' + t\rho'^2) + (\rho + zq - cq\lambda)\rho'' = 0.\end{aligned} \tag{5}$$

By means of (3) and (4), it is found that this may be written as

$$\begin{aligned}(a\lambda - cz)\lambda'' + \left[\frac{1}{z^2 - a\rho^2}\right][(z - \rho q)\rho' - \rho p]^2 \\ + (z - c\lambda)(r + 2s\rho' + t\rho'^2) + (\rho + zq - cq\lambda)\rho'' = 0.\end{aligned} \tag{6}$$

From (1) and (2), it is seen that the great circles of the sphere S are defined by

$$A\rho\cos\theta + B\rho\sin\theta + C(z - c\lambda) = 0. \tag{7}$$

After eliminating (A, B, C) by differentiation, we find

$$r + 2s\rho' + t\rho'^2 + q\rho'' - c\lambda'' - \frac{2\rho'}{\rho}(p + q\rho' - c\lambda') + (z - c\lambda)\left(1 + \frac{2\rho'^2}{\rho^2} - \frac{\rho''}{\rho}\right) = 0. \tag{8}$$

By (3), (4), and (6), this simplifies into the form

$$\frac{\lambda}{\rho}(z - \rho q)\rho'' - \lambda(r + 2s\rho' + t\rho'^2) + \left[\frac{c}{z^2 - a\rho^2}\right][(z - \rho q)\rho' - \rho p]^2 - \frac{2\lambda\rho'}{\rho^2}[(z - \rho q)\rho' - \rho p] + c\rho^2 - z\lambda = 0. \tag{9}$$

The linear-element $d\sigma$ of the surface $\Sigma: z = f(\rho, \theta)$, is

$$d\sigma^2 = (\rho^2 + p^2)d\theta^2 + 2pqd\theta d\rho + (1 + q^2)d\rho^2. \tag{10}$$

The geodesics are given by the equation

$$[\rho^2(1 + q^2) + p^2]\rho'' = (p\rho' - \rho^2 q)(r + 2s\rho' + t\rho'^2) + \rho[2(1 + q^2)\rho'^2 + 3pq\rho' + (\rho^2 + p^2)]. \tag{11}$$

To discover the common geodesics, we have to eliminate ρ'' between (9) and (11). This leads to a cubic equation in ρ'. As a consequence, we may state the following result:

In general, there are at most $3\infty^1$ geodesics of a surface Σ which are projected into great circles of a sphere S under a perspectivity **[4]**.

Let more than $3\infty^1$ geodesics be projected into great circles. The cubic equation in ρ' must be identically zero. Upon placing the coefficient of ρ'^3, equal to zero, we find

$$\lambda(z - \rho q)pt = 0. \tag{12}$$

Neither λ nor $(z - \rho q)$ can be zero. For otherwise we merely find a cone, real or imaginary, with vertex at the origin. In this case, the perspectivity is degenerate.

Therefore in all cases, we must have $pt = 0$.

5. Part II of the proof. *The only surfaces Σ which satisfy the conditions of our fundamental theorem and for which $p \neq 0$ are planes.* By (12), we find $t = 0$ Hence any such surface Σ must be of the form

$$z = \rho\alpha(\theta) + \beta(\theta);\quad p = \rho\alpha_\theta + \beta_\theta \neq 0,\ q = \alpha,\ r = \rho\alpha_{\theta\theta} + \beta_{\theta\theta},\ s = \alpha_\theta,\ t = 0. \tag{13}$$

Upon setting the coefficients of ρ'^2 in (9), where ρ'' is given by (11), equal to zero, and noting that $z-\rho q \neq 0$, we obtain

$$2\lambda p(s\rho - p)(z^2 - a\rho^2) + c\rho^2(z - \rho q)[p^2 + \rho^2(1 + q^2)] = 0. \tag{14}$$

First of all, let us suppose that λ is rational. The discriminant $z^2-a\rho^2 = (\alpha^2-a)\rho^2+2\alpha\beta\rho+\beta^2$, of the quadratic equation (3), must be a perfect square in ρ. This is so only when $a\beta=0$. But if $\beta=0$, the surface is a cone with vertex at the origin. Hence $\beta\neq 0$ and it follows that $a=0$, that is, $c^2=1$. This means that the sphere S passes through the origin. Since $c^2=1$, it follows by (3) and (13) that the condition (14) may be written in the form

$$\begin{aligned}\beta_\theta(\alpha\rho+\beta)(\alpha_\theta\rho+\beta_\theta)[(\alpha^2+1)\rho^2+2\alpha\beta\rho+\beta^2] \\ -\beta\rho^2[(\alpha_\theta^2+\alpha^2+1)\rho^2+2\alpha_\theta\beta_\theta\rho+\beta_\theta^2]=0.\end{aligned} \tag{15}$$

This is an identity in ρ. Upon setting the term independent of ρ equal to zero, we have $\beta^3\beta_\theta^2=0$. Since $\beta\neq 0$, we find that $\beta=z_0$, a non-zero constant.

The preceding equation reduces to the condition, $\alpha_\theta^2+\alpha^2+1=0$. Thus we obtain the solutions, $z=i\rho\cos(\theta-\alpha)+z_0$, where α is a constant. In cartesian coordinates, these are the imaginary planes, $z=i(x\cos\alpha+y\sin\alpha)+z_0$, tangent to the null sphere, $x^2+y^2+(z-z_0)^2=0$. All the straight lines of any one of these planes are not the perspective images of all the great circles of our sphere S with respect to the origin. Thus the case where λ is rational does not yield any solutions whatsoever.

The only case that remains to be considered in this section is where λ is irrational. By (14), we deduce the conditions

$$s\rho - p = 0, \qquad c[p^2 + \rho^2(1 + q^2)] = 0. \tag{16}$$

From (13), we find that $\beta=z_0$, a non-zero constant. If $c\neq 0$, we get the imaginary planes discussed above. Hence it follows that $c=0$ and $\beta=z_0\neq 0$. Of course, this means that the sphere S must have its center at the origin.

Upon setting both the coefficient of ρ' and the term independent of ρ' in the cubic identity obtained by eliminating ρ'' from (9) and (11) equal to zero, we obtain

$$\alpha_\theta(\alpha_{\theta\theta}+\alpha)=0, \qquad \alpha(\alpha_{\theta\theta}+\alpha)z_0+\rho(\alpha_{\theta\theta}+\alpha)(\alpha_\theta^2+\alpha^2+1)=0. \tag{17}$$

The second is an identity in ρ. From it, we deduce that all possible solutions satisfy the condition, $\alpha_{\theta\theta}+\alpha=0$. Hence all the solutions are: $z=\rho(A\cos\theta+B\sin\theta)+z_0$. These are all the planes, $z=Ax+By+z_0$.

Therefore we have succeeded in proving that the gnomic projections are the only ones which satisfy the conditions of our theorem and for which $p\neq 0$.

6. Part III of the proof. *The only solutions of our problem for which $p=0$, are the spheres (and planes $z=a$ constant) homothetic to the sphere S with respect to the point of perspectivity.* Since $p=0$, the cubic identity in ρ' obtained as a result

of eliminating ρ'' from equations (9) and (11), yields the equations

$$\begin{aligned} t\lambda(zq+\rho)(z^2-a\rho^2) &= c\rho(1+q^2)(z-\rho q)^2, \\ q\lambda(zq+\rho) &= c\rho^2(1+q^2). \end{aligned} \tag{18}$$

Either of the conditions $\lambda=0$, or $z^2=a\rho^2$, leads to a cone (real or imaginary), with vertex at the origin. Henceforth $\lambda\neq 0$ and $z^2-a\rho^2\neq 0$.

If $1+q^2=0$, then $zq+\rho=0$. This again yields the imaginary cone with vertex at the origin. Thus $1+q^2\neq 0$.

If $c=0$ and $zq+\rho\neq 0$, then $q=0$. In this case, the sphere S has its center at the origin and our surfaces Σ are the planes, $z=$a constant. Again we have obtained a gnomic projection.

If $c=0$ and $zq+\rho=0$, the sphere S has its center at the origin and the surfaces Σ are spheres concentric with S.

The remaining case to be discussed is that in which $\lambda\neq 0$, $z^2-a\rho^2\neq 0$, $1+q^2\neq 0$, $c\neq 0$, and of course $t=0$ From (18), we derive the equation

$$\frac{t}{q}(z^2-a\rho^2)=\frac{1}{\rho}(z-\rho q)^2. \tag{19}$$

To solve this equation, introduce the substitutions

$$\rho^2+\phi(z)=0, \qquad q=-\frac{2\rho}{\phi_z}, \qquad t=-\frac{2}{\phi_z}+\frac{4\phi\phi_{zz}}{\phi_z^2}. \tag{20}$$

Substituting these into (19), we find

$$2(z^2+a\phi)\phi_{zz}-a\phi_z^2-4z\phi_z+4\phi=0. \tag{21}$$

Differentiating this with respect to z, we get $\phi_{zzz}=0$. Thus ϕ is a quadratic polynomial in z. Substitute this into (21). By (20), we find that our surfaces Σ must necessarily be of the form

$$\rho^2+\alpha z^2+2\beta z+\gamma=0, \tag{22}$$

where (α, β, γ) are constants satisfying the condition

$$\gamma(a\alpha+1)-a\beta^2=0. \tag{23}$$

Substitute (22) into the second of equations (18). We find that

$$[z(1-\alpha)-\beta]\lambda=c[(\alpha-1)(\alpha z^2+2\beta z)+(\beta^2-\gamma)]. \tag{24}$$

It is noted that the conditions $\alpha=1$, $\beta=0$, can not hold simultaneously, for otherwise $\gamma=0$, and we would get an imaginary cone with vertex at the origin.

Since the coefficient of λ in (24) can not be identically zero, we can solve for λ and substitute the result into the quadratic equation (3). This yields the following identity of fourth degree in z:

$$
\begin{aligned}
&ac^2[(\alpha-1)(\alpha z^2+2\beta z)+(\beta^2-\gamma)]^2 \\
&\quad +2c^2z[z(\alpha-1)+\beta][(\alpha-1)(\alpha z^2+2\beta z)+(\beta^2-\gamma)] \\
&\quad -[z(\alpha-1)+\beta]^2[(\alpha-1)z^2+2\beta z+\gamma]=0.
\end{aligned} \tag{25}
$$

We shall prove that $\alpha=1$. Let us suppose the contrary. Substitute $z=-\beta/(\alpha-1)$ into the preceding equation. We find $ac^2[\beta^2-(\alpha-1)\gamma]=0$. If $a\neq 0$, we obtain by this and from (23) that $\beta=\gamma=0$. Thus we have found a cone with vertex at the origin. Hence if $\alpha\neq 1$, then $a=0$. By (23), we deduce that $\gamma=0$. If $\alpha\neq 1$, we have $a=0$ and $\gamma=0$. Thus we can divide out of (25) the factor $z\,[z(\alpha-1)+\beta]$. The result is

$$
2[(\alpha-1)(\alpha z^2+2\beta z)+\beta^2]-[z(\alpha-1)+\beta][(\alpha-1)z+2\beta]=0. \tag{26}
$$

Upon placing the coefficient of z in this, equal to zero, we find $(\alpha-1)\beta=0$. Since $\alpha\neq 1$, we see that $\beta=0$. Thus $\beta=\gamma=0$ and we have found a cone with vertex at the origin.

The above argument shows that $\alpha=1$ in all cases. Since $\alpha=1$, we find from (23) that the value of γ is

$$
\gamma=\frac{a\beta^2}{c^2}. \tag{27}
$$

Upon substituting $\alpha=1$ and this value of γ into (25), we find that (25) is identically zero. Thus by (22), we discover that our surfaces Σ must be the spheres

$$
x^2+y^2+(z+\beta)^2=\frac{\beta^2}{c^2}, \tag{28}
$$

homothetic to our sphere S. Moreover by (25), the magnifying function λ is the constant $\lambda=-\beta/c$.

7. Conclusion. The above completes the proof of our fundamental theorem. The distinction between the result on conformal perspectivities and that of geodesic perspectivities should be noticed. In a conformal perspectivity, a point on the sphere S can correspond to either one of the two images on the homothetic sphere Σ, whereas in our new theorem, a point on the sphere S can correspond only to the single homothetic image on the second sphere Σ.

References

1. Kasner and De Cicco, Conformal perspectivities upon a sphere. This is a forthcoming paper in the Revista de Matematecas de la Universidad Nacional de Tucuman, Argentina. In the same volume, see an interesting extension to differential projective geometry by Terracini.

2. Kasner and De Cicco, Converse of Ptolemy's theorem on stereographic projection, Proceedings of the National Academy of Sciences, Volume 31, 1945.

3. Kasner and De Cicco, Converse theory of gnomic and equiareal perspectivities, Proceedings of the National Academy of Sciences, Volume 32, 1946, pp. 16–19.

4. Kasner, The problem of partial geodesic representation, Transactions of the American Mathematical Society, Volume 7, 1906, pp. 200–206.

APPROXIMATING SUMS

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1. Introduction. The approximating sums for the definite integral of a function may be formed largely without reference to the character of the path. We propose to determine a necessary and sufficient condition for their convergence in the case of integrands which are analytic functions of a complex variable.

2. Necessary conditions. We denote the interval $0 \leqq t \leqq 1$ by I; any finite sequence $t_0, \cdots, t_n$, with $0=t_0 \leqq t_1 \leqq \cdots \leqq t_n=1$, determines a composition Δ. We write $\Delta t_i = t_i - t_{i-1}$, $i=1, \cdots, n$, and $|\Delta; t| = maximum_{i=1,\cdots,n}\ \Delta t_i$ (the *mesh* of Δ). A path is given by an arbitrary complex function $g(t)$ on I; the point set $g(I)$ we denote by C. We define z_i as $g(t_i)$, for a given path and decomposition, and Δz_i as $z_i - z_{i-1}$, $i=1, \cdots, n$; we shall use the symbol $|\Delta; z|$ for the maximum of the numbers $|\Delta z_i|$. Summation from 1 to n, the latter being determined by Δ, is indicated by $\sum$; it will be useful to have a special symbol for $\sum |\Delta z_i|^2$, for which we introduce $\|\Delta; z\|$. Intermediate points z_i' on the path are obtained by forming $g(t_i')$, with $t_{i-1} \leqq t_i' \leqq t_i$, for $i=1, \cdots, n$.

The *admissible* integrands will be single-valued complex functions $f(z)$, defined and differentiable on an open set G which contains the set C; the set G may vary with the integrand.

As usual, we say that $\lim \sum f(z_i')\Delta z_i$ exists, if there is a complex number J such that for $e>0$, a $d>0$ exists, with $|\sum f(z_i')\Delta z_i - J| < e$ whenever $|\Delta; t| < d$.

We state first the following lemma.

LEMMA 1. *A necessary and sufficient condition for the existence of the limit of the approximating sums $\sum f(z_i')\Delta z_i$ is the existence of the limit for special intermediate points, where each t_i' has one of the values t_{i-1} or t_i.*

For the proof, it suffices to point out that by virtue of the identity

$$\sum f(z_i')(z_i - z_{i-1}) = \sum f(z_i')(z_i - z_i') + \sum f(z_i')(z_i' - z_{i-1}),$$

a general approximating sum is equal to a special approximating sum with a mesh which is certainly not larger.

THEOREM 1. *In order that the limit of the approximating sums exists for the (admissible) integrand $f(z)=z$ it is necessary that* $\lim \sum |\Delta z_i|^2 = 0 \equiv \lim \|\Delta : z\| = 0$.

With a subset S of the set $1, \cdots, n$ we associate the special selection: $z_i' = z_{i-1}$ if i is in S; otherwise, $z_i' = z_i$. We shall presently attach, in specified ways, to each Δ a set $S(\Delta)$; summation over this subset will be indicated by $\sum'$.

The difference between $\sum f(z_i)\Delta z_i$ and $\sum f(z_i')\Delta z_i$ becomes, in our case, $\sum z_i \Delta z_i - \sum z_i' \Delta z_i = \sum (z_i - z_i')\Delta z_i = \sum' (\Delta z_i)^2$. If the limit of the approximating sums exists, this sum must tend towards zero; and the same will hold for its real and imaginary parts, $\sum' \mathrm{Re}((\Delta z_i)^2)$ and $\sum' \mathrm{Im}((\Delta z_i)^2)$.

We now choose $S(\Delta)$, in four ways; namely, as the set of indices for which respectively $\mathrm{Re}((\Delta z_i)^2) > 0$, $\mathrm{Re}((\Delta z_i)^2) < 0$, $\mathrm{Im}((\Delta z_i)^2) > 0$, and $\mathrm{Im}((\Delta z_i)^2) < 0$.

Combination of the resulting limit relations yields

$$\lim \sum [\,|\,Re((\Delta z_i)^2)\,| + |\,\mathrm{Im}((\Delta z_i)^2)\,|\,] = 0;$$

since the terms of this sum dominate $|(\Delta z_i)|^2$, we obtain the desired result: $\lim \sum |\Delta z_i|^2 = 0$.

An immediate corollary is the following.

LEMMA 2. *If the limit of the approximating sums exists for all admissible integrands, the function $g(t)$ is continuous.*

When the quantity $|\Delta; z| \equiv \max_{i=1}^{n} |\Delta z_i|$ is less than or equal to the square root of $\|\Delta; z\|$, it must tend toward zero with $|\Delta; t|$. Since every t-interval of I can be contained in a decomposition whose mesh is the length of the interval, we obtain uniform continuity of $g(t)$. The set C will thus have to be compact; this we shall use in the next section.

3. Sufficient conditions. Turning to the sufficiency question, we verify first that for each admissible $f(z)$ and each compact set C there is an $r>0$ and an M such that for any p on C the inequality $|z-p| \leqq r$ implies that z is in G and that $|f(z)| \leqq M$. For if this were not so, we could produce, for $i=1, 2, \cdots$, points p_i, z_i with p_i on C, $|p_i - z_i| \to 0$, and either z_i outside of G or $|f(z_i)| > i$. Because of the compactness of C we could then arrange it so that the sequence p_i converges to a point p of C; the sequence z_i would have to converge to the same point. Since G is open, the points z_i lie ultimately in G; since $f(z)$ is continuous at p, the numbers of $f(z_i)$ will approach $f(p)$ and their absolute values will stay bounded.

So far we have used only the continuity of the integrand; let us now make use of the property of differentiability. For each p on C we consider the function $(f(z)-f(p))/(z-p)$. It is bounded near p and may therefore be continued into p, by the theorem on removable singularities; we thus obtain a function which is differentiable for $|z-p| \leqq r$. For $|z-p| = r$, its absolute value is at most $2M/r$; since such a function has its maximum on the boundary, we get $f|(z)-f(p)| \leqq 2M/r$ for $|z-p| \leqq r$. Thus we have established the following result.

LEMMA 3. *For a compact set C an admissible integrand has the following property, (property X): there exist an $r>0$ and an L such that for p on C, $|z-p| < r$ implies $|f(z)-f(p)| \leqq 2L|z-p|$.*

In the next lemma we employ the Cauchy integral (along straight segments or polygonal lines, for continuous integrands). From the properties of admissible integrands only the continuity and property X come into play. The set C is again assumed to be compact.

LEMMA 4. *If a function $f(z)$ is continuous on G and has property X, then $|\Delta; z| < r$ implies*

$$\left| \sum f(z_i)\Delta z_i - \sum \int_i f(z)dz \right| \leqq L\|\Delta; z\|. \tag{1}$$

Here the integral $\int_i$ is understood to be taken along the straight segment from z_{i-1} to z_i; it exists since the integrand is continuous there. For the sum of the integrals, which is the integral along a polygonal path, we write $\int_\Delta$.

For the proof, we consider $f(z_i)\Delta z_i - \int_i f(z)dz$, which is equal to $\int_i (f(z_i) - f(z))dz$. Using property X we obtain

$$\left| \int_i (f(z_i) - f(z))dz \right| \leqq \int_i |f(z_i) - f(z)|\,d|z|$$

$$\leqq 2L \int_i |z_i - z|\,d|z| = L|\Delta z_i|^2.$$

The inequality (1) is an immediate consequence.

We pass on to an observation which is essentially a matter of elementary topology in the plane; the approximating sums are not directly involved.

LEMMA 5. *If the function $g(t)$ is continuous and the integrand $f(z)$ admissible, then there exists an $e>0$ such that all integrals $\int_\Delta f(z)dz$ with $|\Delta; t| < e$ are equal.*

To prove this we choose e so small that $|\Delta; t| < e$ implies $|\Delta; z| < r$. This is possible since $g(t)$ is uniformly continuous. Since two decompositions of mesh $<e$ have a common subdecomposition of mesh $<e$, it suffices to show that passing to a subdecomposition does not change the integral. This in turn will be taken care of if we can do it in the special case where only one new intermediate value t^* is inserted between (say) t_i and t_{i-1}. We have then $|t^* - t_{i-1}| < e$ and $|t^* - t_i| < e$, thus also $|z^* - z_{i-1}|$, $|z^* - z_i| < r$. These inequalities show that the triangle z_{i-1}, z^*, z_i lies in the circle $|z^* - z| < r$, in which $f(z)$ is differentiable.

The difference between the two integrals is readily expressed as an integral over the boundary of the triangle; it vanishes, by Cauchy's integral theorem, and this completes the proof of the lemma.

Let us designate by J the common value—or limit—of these integrals $\int_\Delta$ for $|\Delta; t| < e$. From (1) we obtain

$$|\sum f(z_i)\Delta z_i - J| \leqq L\|\Delta; z\|, \tag{2}$$

which is valid whenever $|\Delta; t| < e$.

What happens if we choose other intermediate points? We may, as was pointed out in Lemma 1, restrict ourselves to the case where the intermediate points are special; that is, $z_i' = z_i$ or z_{i-1}. For $|\Delta; t| < e$, the property X of admissible integrands yields if $g(t)$ is continuous, the inequality

$$|\sum f(z_i)\Delta z_i - \sum f(z_i')\Delta z_i| \leqq 2L\|\Delta; z\|. \tag{3}$$

Indeed, the left member does not exceed $\sum |f(z_i) - f(z_{i-1})|\,|\Delta z_i|$, which, if $|\Delta; z| < r$, is dominated by $\sum 2L|z_i - z_{i-1}|\,|\Delta z_i| = 2L\|\Delta; z\|$.

The combination of (2) and (3) yields

$$|\sum f(z_i')\Delta z_i - J| \leqq 3L\|\Delta; z\| \tag{4}$$

which is seen to hold if (I) $g(t)$ is continuous, (II) the integrand is admissible, and (III) $|\Delta; t| < e$.

4. Recapitulation. In view of Theorem 1, Lemmas 1 and 2, and equation (4), we may finally present the answer to our question:

THEOREM 2. *A necessary and sufficient condition for the convergence of the approximating sums belonging to all admissible integrands is that* $\lim \|\Delta; z\|$ *be equal to zero.*

ILLUSTRATIVE EXAMPLE. The path defined by $g(t) = t + it \sin(1/t)$, $0 < t \leqq 1$, $g(0) = 0$, satisfies $\lim \|\Delta; z\| = 0$, without being of bounded variation.

5. Further questions. We have tried to present our result in an elementary form, keeping at a level which might be attained in a first course in complex variable. We have not given the name integral to the limit of the approximating sums, since doing so would perhaps involve the obligation of developing the properties of this limit.

In conclusion I mention that D. W. Hall once asked me a related question about the integral along Jordan arcs. Moreover, I suggest the following additional questions: What are the properties of the paths with $\lim \|\Delta; z\| = 0$? To what extent is the stronger form of Cauchy's theorem preserved? Must new conditions be imposed if $f(z)$ satisfies merely a Lipschitz condition? For what paths do decompositions exist with arbitrarily small $\|\Delta; z\|$? What about the limit of the approximating sums as $\|\Delta; z\| \to 0$?

A NECESSARY AND SUFFICIENT CONDITION OF WIENER

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1. Introduction. A necessary and sufficient condition, that an object belong to a certain defined class, is a tautology in the sense that it is equivalent, from the point of view of logical analysis, to the original proposition. A sufficient condition, which is not necessary, offers apparently an immediate advantage in that its purpose is to point out a recognizable subclass of objects for which the given definition is valid. Similarly, a necessary condition, which is not sufficient, excludes a recognizable subclass which does not contain the defined objects. These one-sided conditions therefore serve to delimit workable situations. But unless we assume a mystical attitude toward mathematics, so that we remain in a continual state of surprise, the discovery of an essential application of a necessary and sufficient condition should strike us with more force than the usual discovery of mathematical fact—or else, we should realize that the concreteness of mathematical fact is as important as logical structure.

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2. Regular points and Wiener's condition. An interesting case of this kind is an application of Wiener's necessary and sufficient condition that a point be a *regular boundary point* with respect to a given domain and Laplace's equation.* It has been remarked that Wiener's condition is "beautiful," presumably because it analyzes the neighborhood of the boundary point in an appropriate way.

For domains with sufficiently smooth boundaries we know that there is one and only one function which is a solution of Laplace's equation within the domain and takes on the values of a given continuous function for arbitrary approach to the boundary. An arbitrary domain can be approximated arbitrarily closely by a domain with such a smooth boundary; indeed, an arbitrary domain T is the limit as $n\to\infty$ of a sequence of such smooth boundary domains T_n, each domain of the sequence strictly containing the previous one.

Given a function f, which is continuous on the boundary, it can be extended so as to be continuous within a certain distance of the boundary. There will then correspond to each approximating domain a harmonic function u_n which belongs to the given continuous function, taking on its values on the boundary of T_n. As $n\to\infty$, the function u_n tends to a function u which is harmonic in the domain T; the function u depends merely on f and not on its extension. The question remains as to how this sequence solution u behaves on approach to a point p of the boundary of T. The point p is a *regular boundary point* for T if, given arbitrarily the continuous boundary function f, we have

$$\lim_{P\to p} u(P) = f(p), \qquad \text{for } P \text{ in } T.$$

Lebesgue showed that if p is the vertex of a sufficiently sharp spine reaching into T, p is not regular. Lebesgue also gave a necessary and sufficient condition that p be a regular boundary point in terms of the existence of a special function called a *barrier*, which is a harmonic function (or a harmonic function plus a potential of positive mass) with certain defined properties. To be exact, the barrier has to have a positive lower bound in T, no matter how small a neighborhood of p is excluded, and has to tend to 0 at p. One use of this condition is the concreteness of its application as a sufficient condition; for example, it can be shown by this means or by Wiener's condition that if p is the vertex of a small triangle which lies outside of T, then p is regular.

Wiener's condition depends on the notion of the *capacity* of a bounded closed set in space; for example, the set might be a piece of conducting surface or a solid conductor. The capacity is defined as the upper bound (least upper bound) of the total positive mass which can be distributed on the set so that the potential of the distribution nowhere exceeds the value 1. The capacity of a sphere is equal to its radius while that of a circular disc is $(2/\pi)$ times its radius. Capacity is of dimension one in length. If we have two sets F' and F, such that to each point q' of F' there corresponds a point q of F, to distinct points q', r'

* N. Wiener, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3, 1924, pp. 24–51 and pp. 127–146.

of F' there correspond distinct points q, r of F, and the respective distances satisfy $|q'r'| \leqq |qr|$, then the capacity of F' is not greater than the capacity of F. In particular, this condition is satisfied if F' is contained in F. If the capacity of F is positive there exists mass, equal to the capacity of F (whose distribution we may call a conductor distribution), of which the potential (the conductor potential) takes on the value 1, but does not exceed that value. The conductor potential takes on the value 1 at all regular boundary points of the infinite domain whose boundary is the exterior frontier of F.

Now let p be a boundary point of a given domain T, and λ a positive number less than unity. Let γ_k be the capacity of the closed set e_k of points q which do not belong to T and which satisfy the condition

$$\lambda^k \leqq \text{distance } pq \leqq \lambda^{k-1}.$$

Consider the series

$$\sigma = \sum_1^\infty \left(\frac{\gamma_k}{\lambda^k}\right).$$

Then p is regular if σ diverges, but is irregular if σ converges. This is Wiener's condition. One can show that Lebesgue's condition is equivalent to it.*

3. Surfaces of minimum capacity. Let us now consider the application. We know that in three dimensions there exist harmonic functions bounded over all space provided that we allow them to be multiple-valued. The multiplicity may be any finite value; in particular the function may be double-valued. It is this latter case which we are to consider. Corresponding to branch points in the plane we have branch curves in space. We wish to investigate the continuity of the double-valued function on these branch curves.

For example, the conductor potential of a circular disc is given by the formula

$$V(P) = 1 - \frac{2}{\pi} \arctan \frac{\sqrt{\mu}}{a},$$

where μ is the ellipsoidal parameter of the point P in the system of spheroidal coordinates for which the level surface corresponding to μ degenerates into the disc when $\mu = 0$. This potential $V(P)$ can be continued harmonically across the disc in either direction by replacing $\sqrt{\mu}$ by $-\sqrt{\mu}$, and thus yields a double-valued non-negative function, harmonic in all space except on the circumference s of the disc, equal to 0 and 2 respectively at ∞, and the sum of its two values for any P remaining equal to 2. It will be noticed that in terms of this explicit expression, $V(P)$ is continuous on s; in fact, the behavior of $V(P)$ can be examined in detail in the neighborhood of s by introducing coördinates α, β with

$$x^2 + y^2 + z^2 = a^2 + \alpha^2 + 2a\alpha \cos \beta, \qquad z = \alpha \sin \beta,$$

* O. D. Kellogg, Foundations of Potential Theory, Berlin, 1929, pp. 326–334.

so that, writing

$$\frac{x^2+y^2}{a^2+\mu}+\frac{z^2}{\mu}=1,$$

μ will be given by the expression

$$\mu=\left(\frac{\alpha}{2}\right)\{2a\cos\beta+\alpha+[4a^2+\alpha^2+4a\alpha\cos\beta]^{1/2}\}.$$

The potential $V(P)$ has the value 1 on the disc and on s. The disc plays the role of a cut, s being the branch curve.

This analogy to Riemann surfaces was used by Sommerfeld* for the solution of a number of problems and has been noticed and applied by Courant, the present author, and others. Let us suppose that we have a closed curve s in space, itself of zero capacity, such that the points of the interior of some sphere which contains it may be put into one to one correspondence with the points of a second sphere in such a way that s corresponds to a circle inside the second sphere. We are able to show that there exists a particular surface cap S, bounded by s, such that, among all such caps, S has minimum capacity. The cap S is in fact the level surface $V=1$ of a uniquely determined double-valued function which is bounded in all space, harmonic except on s, and equal to 0 and 2, respectively, at ∞. The existence and properties of S are obtained in terms of V. The surface S separates no points from ∞. If S is made a cut surface, the branch V_1 of V which vanishes at ∞ is the conductor potential of S. At any point the sum of the two branch values remains equal to 2.†

Every point of s is a limiting point of S, and of the infinite domain complementary to S, and every small closed curve which loops s cuts S. It is not true, however, that V is necessarily continuous on s, taking on the value 1 for arbitrary approach to a point of s. This statement is easily verified in the special case of the plane curve s obtained by adding a thin flat spine to the disc which we have discussed above.

THEOREM: *It is true that V is continuous at any point p of s where s has a tangent line, or where s has a forward and a backward tangent which do not make a zero angle.*

In order to prove this theorem let t_1 be the forward tangent at p, t_2 the backward tangent and π a sector of a half-plane bounded by t_1 and t_2. Let F be the set of points, in a closed spherical neighborhood Γ of p, which belong to $S+s$. Draw two thin single-sheeted cones C_1 and C_2, with axes t_1 and t_2, respectively, and common vertex p, cutting the boundary of Γ in small circles which are exterior to each other and separated; the radius of Γ may be taken small enough

* A. Sommerfeld, Proceedings of the London Mathematical Society, vol. 28, 1897, pp. 395–429. H. Bateman, Partial Differential Equations of Mathematical Physics, New York, 1944, p. 466.

† G. C. Evans, Bulletin of the American Mathematical Society, vol. 47, 1941, pp. 717–733.

so that within Γ the curve s lies entirely inside the cones.

We now project certain of the points of F circularly to the half plane π, in order to form a set F'. Consider the set of circles C in Γ with centers on t_1+p, in planes orthogonal to t_1, whose discs do not contain any points interior to C_2 but contain all the points of their plane sections with C_1. The circumferences C loop s, and therefore each C contains a point of F. By rotation along C the point is projected into a point of the half plane π, and these projected points fill a certain sector with a non-zero angle at p. This plane sector plus the point p constitute the set F'.

To each point of F' corresponds, by the projection, one or more of the points of F, of which one may be chosen. To distinct points q_1', q_2' of F' correspond distinct points q_1, q_2 of F, since q_1', q_2' lie on distinct circumferences C. Moreover, with regard to the respective distances, $|q_1'q_2'| \leqq |q_1q_2|$. In fact, if we take cylindrical coordinates (r, θ, z), t_1 being the axis of z, we find that

$$|q_1'q_2'|^2 = (r_2 - r_1)^2 + (z_2 - z_1)^2, \quad r_1^2 = x_1^2 + y_1^2, \quad r_2^2 = x_2^2 + y_2^2$$

$$|q_1q_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$|q_1q_2|^2 - |q_1'q_2'|^2 = 2(r_1r_2 - x_1x_2 - y_1y_2) = 2r_1r_2[1 - \cos(\theta_2 - \theta_1)].$$

Hence the capacity of F' cannot exceed the capacity of F.

Similarly, if we consider subsets e_k of F which are specified in Wiener's criterion and the subsets e_k' of F' which lie in the corresponding annular plane regions, each e_k' is a circular projection of e_k. Hence the capacity γ_k' of e_k' cannot exceed the capacity γ_k of e_k. Accordingly, the series σ will diverge if the series

$$\sigma' = \sum_1^\infty \frac{\gamma_k'}{\lambda^k}$$

diverges.

But F' contains a plane triangle with vertex at p. And as was mentioned above, p is a regular point for the infinite domain whose sole boundary is this plane triangle. It follows, therefore, using the necessary part of the Wiener condition, that the Wiener series for this triangle diverges. But each term of this series is dominated by the corresponding term in σ', since each annular portion of the triangle is contained in the corresponding e_k' Hence σ' diverges and σ diverges. Accordingly, using the sufficiency part of the Wiener condition, p is a regular boundary point for the infinite domain T of which S is the sole boundary, and

$$\lim_{P \to p} V_1(P) = 1, \qquad P \text{ in } T.$$

Also, if $V_2(P)$ is the other branch of the function V, since $V_1(P)+V_2(P) \equiv 2$, we have $\lim_{P \to p} V_2(P) = 1$. In other words, $V(P)$ is continuous on s.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Send material for this department directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

ON THE NEED FOR CARE IN USING A CERTAIN INTEGRAL FORMULA

A. E. TAYLOR, University of California, Los Angeles

The formula

$$(1) \qquad \int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \frac{\sqrt{a^2-b^2}\tan \frac{1}{2}x}{a+b}, \qquad a^2 > b^2,$$

is given in many standard integral tables. It must be used with care, however, as the following paradox shows.

Let us take $a=5$, $b=-3$, and calculate the value of the definite integral

$$(2) \qquad \int_0^{3\pi/2} \frac{dx}{5-3\cos x}.$$

The integrand in (2) is positive and continuous for all values of x, so the definite integral is certainly positive. However, if we use (1), we obtain a negative result, namely

$$\int_0^{3\pi/2} \frac{dx}{5-3\cos x} = \tfrac{1}{2} \tan^{-1} (2 \tan \tfrac{1}{2}x) \Big|_0^{3\pi/2} = -\tfrac{1}{2} \tan^{-1} 2.$$

We have here an apparent failure of the fundamental theorem of calculus. If the function on the right in (1) be denoted by $F(x)$, it is true that

$$(3) \qquad F'(x) = \frac{1}{a+b\cos x}$$

for all values of x for which $F(x)$ is defined. But $F(x)$ is not defined when x is an odd multiple of π. Furthermore, if $x_n=(2n+1)\pi$, $n=0, \pm 1, \cdots$, the limits of $F(x)$ at $x=x_n$ from the right and left, respectively, are

$$(4) \qquad F(x_n+0) = \frac{-\pi}{\sqrt{a^2-b^2}}, \qquad F(x_n-0) = \frac{\pi}{\sqrt{a^2-b^2}}$$

if $a+b>0$. If $a+b<0$, the right members of the two equations (4) must be exchanged. Hence there is no way of defining $F(x)$ at $x=x_n$ so as to make the function differentiable there.* Therefore, in applying (1) to evaluate a definite integral, we must take care that the interval of integration does not contain one of the points $x=x_n$ *in its interior.* If a point $x=x_n$ is at an end of the interval we may still use (1), but we must think of $F(x)$ as being defined at the end of the

* Another example of much the same kind, but less likely to occur in practice, was pointed out recently by C. I. Lubin (this MONTHLY, vol. 53, 1946, p. 586).

interval by its limiting value from within the interval. The correct evaluation of (2), for example, is as follows:

$$\int_0^{\pi} \frac{dx}{5-3\cos x} = F(\pi-0) - F(0) = \frac{\pi}{4} - 0;$$

$$\int_{\pi}^{3\pi/2} \frac{dx}{5-3\cos x} = F\left(\frac{3\pi}{2}\right) - F(\pi+0) = -\tfrac{1}{2}\tan^{-1} 2 + \frac{\pi}{4};$$

$$\int_0^{3\pi/2} \frac{dx}{5-3\cos x} = \frac{\pi}{2} - \tfrac{1}{2}\tan^{-1} 2 = 1.02.$$

An example of the occurrence of the foregoing situation in practice is furnished by the following problem. Consider an infinitely long cylindrical surface $x^2+y^2=a^2$. Let the two halves of this surface for which $y>0$ and $y<0$, respectively, be conductors which are insulated from each other and given potentials 1 and -1, respectively. The potential along the line with cylindrical coördinates (r, θ), $0 \leqq r < a$, is given by Poisson's integral formula

$$V(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{a^2-r^2}{a^2+r^2-2ar\cos(\phi-\theta)}\,d\phi, \tag{5}$$

where $f(\phi)=1$ when $0<\phi<\pi$, $f(\phi)=-1$ when $\pi<\phi<2\pi$. We make the change of variable $x=\phi-\theta$, and use (1) with care. It is necessary to distinguish three cases. The results are

$$V(r,\theta) = 0 \quad \text{if} \quad \theta = 0 \quad \text{or} \quad \theta = \pi;$$

$$V(r,\theta) = \frac{2}{\pi}\tan^{-1}\left[\frac{a+r}{a-r}\operatorname{ctn}\frac{\theta}{2}\right] + \frac{2}{\pi}\tan^{-1}\left[\frac{a+r}{a-r}\tan\frac{\theta}{2}\right] \mp 1; \tag{6}$$

in the last formula we choose -1 if $0<\theta<\pi$, and $+1$ if $\pi<\theta<2\pi$.

The solution (6) is clumsy in form, and the work of deriving it is tedious. There is a better procedure for the evaluation of the integral (5). We let $x=\phi-\theta$ when $0 \leqq \phi \leqq \pi$, and $y=\phi-\theta$ when $\pi \leqq \phi \leqq 2\pi$. Then

$$V(r,\theta) = \frac{1}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{(a^2-r^2)\,dx}{a^2+r^2-2ar\cos x} - \frac{1}{2\pi}\int_{\pi-\theta}^{2\pi-\theta} \frac{(a^2-r^2)\,dy}{a^2+r^2-2ar\cos y}.$$

In the second integral let $y=\pi+x$, and combine the result with the first integral. In this way we obtain

$$V(r,\theta) = \frac{1}{2\pi}\int_{-\theta}^{\pi-\theta} \frac{4ar(a^2-r^2)\cos x}{(a^2+r^2)^2 - 4a^2r^2\cos^2 x}\,dx. \tag{7}$$

The integral (7) may be evaluated easily by the substitution $u=\sin x$. The final result obtained in this way is much neater than the form (6); it is

$$V(r,\theta) = \frac{2}{\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{a^2-r^2}\right). \tag{8}$$

POLYGONS HAVING A COMMON MEAN

V. O. McBrien, Holy Cross College

1. The following theorem on cyclic n-gons was recently discussed by R. Goormaghtigh, this Monthly, vol. 53, 1946, p. 525:

When two n-sided polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ are inscribed in a circle Γ, the n orthopoles, with respect to $A_1A_2 \cdots A_n$, of all the polygons obtained by omitting one of the vertices of $B_1B_2 \cdots B_n$ and the n orthopoles, with respect to $B_1B_2 \cdots B_n$, of all the polygons obtained by omitting one of the vertices of $A_1A_2 \cdots A_n$ are $2n$ points on a circle having as center the midpoint of the segment joining the orthocenters of the polygons. If ϕ is the angle formed by the mean polygons of the given polygons, the radius of the circle equals that of Γ multiplied by $\cos \frac{1}{2}n\phi$ when n is even and by $\sin \frac{1}{2}n\phi$ when n is odd.

2. It is of interest to discuss the condition for polygons which have a *common mean*, for then the circle reduces to the null circle. The properties of triangles having a common mean have been considered by O. J. Ramler, this Monthly, vol. 47, 1940, p. 140.

Using circular coördinates, let the cyclic n-gon A_i have vertices t_i and the cyclic n-gon B_i have vertices τ_i, where $|t_i| = |\tau_i| = 1$. Designate the symmetric functions of t_i and τ_i by $s_{n,i}$ and $\sigma_{n,i}$, respectively. The orthopole of a variable vertex t_i is given by

$$z = \tfrac{1}{2}(s_{n,1} + \sigma_{n,1}) - \tfrac{1}{2}\left[1 + (-1)^n \frac{\sigma_{n,n}}{s_{n,n}}\right] t_i. \tag{2.1}$$

This is the map equation of the circle of $2n$ orthopoles. The radius of the circle is $\frac{1}{2}\left|\,[1+(-1)^n\sigma_{n,n}/s_{n,n}]\,\right|$. The condition for a null circle is therefore,

$$(-1)^{n-1}\sigma_{n,n} = s_{n,n}. \tag{2.2}$$

For $n=3$, we have $\sigma_{3,3}=s_{3,3}$ which is the condition that a pair of cyclic triangles have a common mean triangle. In general we may say that two cyclic n-gons have a common mean if (2.2) is true. We then have as an important case of the theorem above, the Corollary:

If two cyclic n-gons have a common mean, the 2n orthopoles with respect to the n-gons are coincident.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.

THE TRIAL INTEGRAL METHOD

M. F. SMILEY, Northwestern University

The purpose of this note is to provide a workable and direct explanation* of a method of solution of the non-homogeneous linear differential equation

$$F(D_x)y = f(x), \tag{1}$$

in which $F(u)$ is assumed to be a polynomial in u with real coefficients independent of x. We shall assume that $f(x)$ satisfies a *homogeneous* linear differential equation

$$G(D_x)f(x) = 0, \tag{2}$$

where $G(u)$ is a polynomial in u with real coefficients independent of x.† We insist on real coefficients only because this has been the custom in engineering applications.

We shall suppose that a method of solution of *homogeneous* equations $H(D_x)y=0$, with $H(u)$ a polynomial in u with real coefficients independent of x, has been explained. We feel that this discussion should culminate in a table of the following sort.

Root of $H(u)=0$	Multiplicity	Corresponding terms in the general solution
$u=\alpha$, real	Simple	$Ce^{\alpha x}$
$u=\alpha$, real	μ	$e^{\alpha x}(C_1+C_2x+\cdots+C_\mu x^{\mu-1})$
$u=\alpha\pm\beta i$	Simple	$e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$
$u=\alpha\pm\beta i$	μ	$e^{\alpha x}[\cos\beta x(C_1+C_2x+\cdots+C_\mu x^{\mu-1})+\sin\beta x(C_{\mu+1}+C_{\mu+2}x+\cdots+C_{2\mu}x^{\mu-1})]$

It should be emphasized that the functions $f(x)$ which satisfy our basic assumption are linear combinations of those appearing in this table.

If the polynomial $G(u)$ which occurs in (2) were available, we could reduce (1) to the *homogeneous* equation

* Phillip Franklin's Methods of Advanced Calculus treats this case in a manner quite similar to ours. We claim no originality for the method given here, but we feel that its advantages over the usual method deserve consideration by a teacher of elementary differential equations.

† This is the case in which the well known method of the trial integral applies. See A. Cohen, Differential Equations.

$$G(D_x)F(D_x)y = 0 \tag{3}$$

by applying the operator $G(D_x)$ to (1). The general solution of (3) could then be substituted in (1) and the additional constants introduced by the operator $G(D_x)$ could be evaluated. In fact, a knowledge of the *roots* of $G(u)=0$ and of $F(u)=0$ will suffice to determine the general solution of (3).

To find the roots of $G(u)=0$ which correspond to a given function $f(x)$ we suggest that the table displayed be used *in reverse*. Thus to x^2e^{3x} (or to $(x^2+4)e^{3x}$) there corresponds the roots* (3, 3, 3); while to $xe^x \sin 5x$ (or to $xe^x \cos 5x$) there corresponds the roots $(1\pm 5i,\ 1\pm 5i)$. Having done this, we may then combine the roots of $G(u)=0$ with those of $F(u)=0$ to obtain the general solution of (3) from the displayed table.

Once the trial form of y has been determined, we may proceed to substitute in (1) and evaluate the non-arbitrary constants. Of course, the general solution of (3) will contain the complementary function of (1) and this may be ignored if we desire only a particular integral of (1).

We conclude with a simple example.

$$(D_x^2 + 1)y = x^2 \sin x. \tag{4}$$

Here $F(u)=u^2+1$ and the roots of $F(u)=0$ are $\pm i$. The roots corresponding to $x^2 \sin x$ are $(\pm i,\ \pm i,\ \pm i)$. Hence the roots of $G(u)F(u)=0$ are $(\pm i,\ \pm i,\ \pm i,\ \pm i)$ and

$$y = (A + Bx + Cx^2 + Dx^3)\cos x + (E + Fx + Gx^2 + Hx^3)\sin x.$$

Substitution in (4) yields $D=-1/6$, $B=G=1/2$; $C=F=H=0$; A, E arbitrary. With these values, y is the general solution of (4).

ON THE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

A. B. FARNELL, University of Colorado

Several special equations which are sufficient to illustrate the points in question will be considered here.

For example, consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{ax} \tag{1}$$

and the solution

$$y(x) = c_1e^x + c_2xe^x + \frac{e^{ax}}{(a-1)^2} \tag{2}$$

which satisfies the differential equation formally. The standard approach presented in most elementary texts to this solution is the method of complementary

* We adopt the usual notation in indicating the multiplicity of a root by repetition.

and particular solutions. For a different from 1, the solution is usually left in this form.

Is (2) a solution of (1) for a equal to 1? Most elementary texts infer that in this case (2) is not a solution and has no connection with the solution of (1), and some new approach is needed to find a particular solution; this new approach is usually pulled out of thin air. No cognizance is taken of the fact that c_1 and c_2 are functions of a. If (2) is written more precisely, the need for special considerations for various cases is avoided.

Let $y(0)=y_0$, $y'(0)=y_0'$. Imposing these initial conditions, (2) can be written in the form

$$y(x) = y_0e^x + (y_0' - y_0)xe^x + \frac{e^{ax} - e^x + (1-a)xe^x}{(a-1)^2}. \tag{3}$$

This solution is valid for a different from 1, and as a approaches 1, (3) becomes in the limit

$$y(x) = y_0e^x + (y_0' - y_0)xe^x + \tfrac{1}{2}x^2e^x \tag{4}$$

the correct solution for a equal to 1.

For the differential equation

$$\frac{d^2y}{dx^2} + y = \sin ax$$

the solution can be written in the form

$$y(x) = y_0' \sin x + y_0 \cos x + \frac{\sin ax - a \sin x}{1 - a^2}. \tag{5}$$

For a different from (1), (5) gives the correct solution; and for a equal to 1, the proper solution is

$$y(x) = y_0' \sin x + y_0 \cos x + \tfrac{1}{2}(\sin x - x \cos x), \tag{6}$$

which is obtained from (5) by letting a approach 1.

Although the special methods of approach for various cases mentioned above have the undoubted advantage of reducing the labor involved in solving special equations, examples, such as the ones given above, should help in showing why and when these special methods work.

Finally, consider the differential equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = x.$$

Most students will readily agree that in this case the particular solution y_p might contain x, x^2, and x^3. A little further argument will convince them that y_p should be as general as its derivative and hence should contain a constant term. We thus take y_p in the form

$$y_p = Ax^3 + Bx^2 + Cx + D,$$

which gives the particular solution in every case regardless of the values of the constants in the left-hand side of the differential equation; in one case D might not be determined which means simply that it is arbitrary.

In the same way, for the differential equation

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = x \sin x$$

we have in all cases,

$$y_p = (Ax^2 + Bx + C) \sin x + (Dx^2 + Ex + F) \cos x.$$

After the student is convinced he can form the particular solution in all cases without any special knowledge of the complementary solution, it is then easier to justify the special methods adopted for various cases, and, the reduction in labor by using them will be obvious.

A FUNCTION WITH A FINITE DISCONTINUITY

J. A. WARD, University of Georgia

An illustration of a function with a finite discontinuity that I have found useful in teaching sophomore calculus is the following:

$$y = \text{the degree in } u \text{ of: } (x - 1)u^3 + 5u^2 - 3u + 2.$$

Then y is a discontinuous function of x; for at $x=1$, then $y=2$; but for all other values of x we have that $y=3$. This function seems a little less artificial to sophomores than one of the type $y=(x^2-4)/(x-2)$ which has to be re-defined at $x=2$.

Other examples may readily be constructed along these lines, such as:

$$y = \text{the degree in } u \text{ of } (x - 1)u^{x^2+4} + 5u^2 - 3u + 2.$$

This gives in the xy-plane a parabola with the single discontinuity at $x=1$. Another variation is a continuous function whose derivative has two discontinuities:

$$y = \text{the degree in } u \text{ of } u^{x^3} + 2u^2 - 5u + 3.$$

In the xy-plane this is a parabola with the lower part of the arc discarded and replaced by a chord.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

Correction. Problem E 754 [1947, 39] was improperly credited to C. A. Richmond, Tyngsboro, Massachusetts. The correct proposer is S. T. Thompson, Tacoma, Washington.

E 761. *Proposed by C. R. Perisho, McCook Junior College, Nebraska*

An object with a smooth lower plane surface, and center of gravity h units above this surface, is balanced on a sphere of radius R. Find the relation between h and R which insures stability of the object under small displacements.

E 762. *Proposed by J. R. Van Andel, Naval Air Experimental Station, Philadelphia, Pa.*

Let A_1 and A_2 be two circles with radii a_1 and a_2 and centers $(a_1, 0)$ and $(a_2, 0)$, respectively, with $a_2 > a_1 > 0$. Let C be any circle in the crescent shaped area M between A_1 and A_2, and tangent to both A_1 and A_2.

(a) The locus of the center of C as it sweeps out M is an ellipse with semi-axes $(a_1+a_2)/2$ and $\sqrt{a_1a_2}$.

(b) If C_t is a circle of radius r_t and center $P_t(x_t, y_t)$, where

$$\begin{aligned} r_t &= a_1a_2(a_2 - a_1)\phi_t, \\ x_t &= a_1a_2(a_2 + a_1)\phi_t, \\ y_t &= 2tr_t \\ \phi_t^{-1} &= a_1a_2 + t^2(a_2 - a_1)^2, \end{aligned}$$

then, for any real value of t, C_t lies in M and is tangent to A_1, A_2, and C_{t-1}.

E 763. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenters of the faces of a tetrahedron to the reflections in these faces of the points of intersection of the corresponding altitudes with the circumsphere, are concurrent at the Monge point of the tetrahedron.

E 764. *Proposed by R. J. Walker, Cornell University*

If a_{ij} is the number in the ith row and jth column of a diabolic (pandiagonal magic) square of order five, show that

$$\sum_{i=1}^{5}\left(\prod_{j=1}^{5} a_{ij}\right) = \sum_{j=1}^{5}\left(\prod_{i=1}^{5} a_{ij}\right).$$

This relationship also holds for magic squares of order three. Does it, or any similar relation, hold for magic, or diabolic, squares of any other order—in particular, for diabolic squares of order seven?

E 765. *Proposed by Harold Becker, Omaha, Nebraska*

(a) What is the joint resistance of the general Wheatstone bridge as a function of the five component resistors? (b) What are forms for the components so that they and the resultant will all be integers?

SOLUTIONS

Power Points

E 705 [1946, 36 and 395]. *Proposed by Joseph Rosenbaum, The Milford School, Conn.*

A point in the interior of a simple closed curve, which is nowhere concave, will be called a *power point* if the product of the segments of a variable chord through the point is constant. The existence of how many distinct power points insures that the curve is a circle?

Note by the Proposer. R. A. Rosenbaum points out that this problem is not new, it having appeared as an article entitled *A characteristic property of the circle and sphere; second note*, by K. Yanagihara, in the Tôhoku Mathematical Journal, vol. 11 (1917), p. 55. In the article Yanagihara discusses both this problem and the analogous one for a simply closed surface. His proof for the sufficiency of two power points for the curve is essentially the same as that given by Kelly in the MONTHLY. However, as regards the surface, Yanagihara arrives at the false conclusion that there, too, the existence of two power points is a sufficient condition for the surface to be a sphere. That this is not true is shown by the following example.

Take a circle with a chord AB, and two points M and N on the circle on opposite sides of AB. Next let c be an arc of any plane curve *other than a circle*, passing through M and N and not in the plane of the circle. Now the closed surface generated by the variable circle ABP as P moves from M to N along the curve c is not a sphere, because c is not an arc of a circle. The surface, however, possesses two power points; in fact every point on AB is a power point of the surface, since, by the very manner in which this surface was generated, every section of it by a plane through AB is a circle having AB as a chord.

It is easily shown, however, that the surface above will fail to possess a tangent plane at either A or B, and that the sufficiency of two power points under the additional condition that the surface possesses a unique tangent plane at every point is correct. Since this additional condition is not included in Yanagihara's theorem, his theorem, as stated, is false.

Inasmuch as the existence of three non-collinear power points was mentioned in the editor's note to this problem as being a sufficient condition for a simply closed surface to be a sphere, it may be well to supply the proof for it here.

LEMMA 1. *If a simply closed surface F possesses two power points A and B*

within it, then every point on AB is a power point of F.

This follows immediately from the fact that a simply closed curve possessing two power points is a circle.

LEMMA 2. *If a simply closed surface F possesses three non-collinear power points A, B, C within it, then every point P in the plane ABC is a power point of F.*

Draw any line through P cutting AB and BC in I and J respectively. By Lemma 1, I and J are both power points of F, and since P lies on IJ then, again by Lemma 1, P is a power point of F.

THEOREM. *If a simply closed convex surface F possesses three non-collinear power points A, B, C within it, then F is a sphere.*

The plane ABC cuts F in a circle k. By Lemma 2, every point in this circle is a power point of F. Now consider the sphere S determined by the circle k and a point Q on F outside the circle k. If X is any point on F on the opposite side of the plane of k from Q, then the line QX cuts the plane of k in a point whose power with respect to F is the same as that with respect to S (both being equal to the power of that point with respect to circle k). Hence X must lie on the sphere S. This proves that all points on F which are on the opposite side of the plane of k from Q lie on S. This in turn, by the same argument, leads to the conclusion that every point on F on the opposite side of the plane of k from X lies on the sphere S. Since these two sets of points constitute the entire surface F, the proof is completed.

Consecutive Odd Integers

E 726 [1946, 333]. *Proposed by W. Nicholson, Chicago, Illinois*

If p and N are positive integers, $p>1$, show that N^p is the sum of N consecutive odd integers.

Solution by Murray Barbour, Michigan State College. The formula for the sum of an arithmetic progression is $s=\frac{1}{2}n[2a+(n-1)d]$. Here $s=N^p$, $n=N$, and $d=2$. Substituting these values and solving for a we find $a=N^{p-1}-N+1$. This value for a is necessarily odd, since, for $p>1$, $N(N^{p-2}-1)$ is always even.

Also solved by D. W. Alling, M. Aissen, Joshua Barlaz, Barney Bissinger, W. G. Brady, Paul Brock, W. E. Byrne, M. I. Chernofsky, H. J. Cohen, M. L. Constable, R. E. Crane, Monte Dernham, Benjamin Epstein, J. W. Gaddum, Sydney Glusman, Bernard Greenspan, Stanley Hughart, Meyer Karlin, N. D. Lane, Elmer Latshaw, B. R. Leeds, W. J. LeVeque, C. D. Olds, Margaret Olmsted, K. B. Patterson, C. R. Perisho, C. L. Perry, P. A. Piza, E. D. Schell, E. P. Starke, W. R. Talbot, R. H. Urbano, Alan Wayne, Michael Wilensky, Maud Willey, R. K. Zeigler, and the proposer.

Barlaz proposed the similar but more difficult problem: Let p, N be positive integers, $p>1$. Find conditions on k, a, b so that N^p may be written as the sum of k consecutive terms of the arithmetic progression $ax+b$, $(a, b)=1$, $ax+b>0$.

Dernham pointed out that the problem is a special case of the more general theorem: If M and N are positive integers, $M \geqq N$, then MN is the sum of N consecutive odd or N consecutive even integers, depending upon whether M and N have the same or different parity; namely

$$(M - N + 1) + (M - N + 3) + \cdots \text{ to } N \text{ terms.}$$

Probability of Adjacent King and Jack

E 727 [1946, 333]. *Proposed by Frederick Mosteller, Princeton, N. J.*

What is the probability that a King and a Jack will appear side by side in a shuffled pack?

Solution by N. D. Lane, St. Andrew's College, Ontario. Let us consider the number of ways a King and a Jack will *not* appear together. We may have four, three, two, or none of the Kings side by side.

If no two Kings are side by side, remove the Kings and Jacks from the pack and arrange the other 44 cards in one of 44! different ways. Now the Kings may be inserted in the 45 available places in $P(45, 4)$ ways. The Jacks may now be inserted, one at a time, so that no Jack is next to a King, in $41 \cdot 42 \cdot 43 \cdot 44$ ways. Hence the number of ways no Jack and King will appear side by side when the the Kings are separated is $44!P(45, 4)P(44, 4) = a$, say.

Similarly, if we have two Kings together and the other two separated the number of ways in which a Jack and King will not be together is $44!P(4, 2)P(45, 3)P(45, 4) = b$.

If we have the four Kings arranged in pairs the number is $44!P(4, 2)P(45, 2)P(46, 4) = c$.

If we have three Kings side by side, the number of ways is $44!P(4, 3)P(45, 2)P(46, 4) = d$.

If all four Kings are side by side, the number of ways is $44!P(4, 4)P(45, 1)P(47, 4) = e$.

The probability that a King and a Jack will appear together is then $1-(a+b+c+d+e)/52! = 0.48628$, or almost one-half.

Also solved by H. D. Grossman and John Riordan.

Grossman pointed out that, except for the substitution of 4-spots and 7-spots for Kings and Jacks, this is problem No. 578 of the *National Mathematics Magazine*, April 1945, pp. 367, 368. His solution is the one there presented, and is similar to the above.

Riordan employed the symbolic method (*cf.* Kaplansky, *Symbolic Solution of Certain Problems in Permutations*, Bull. A.M.S., 1944, pp. 906–914). He also showed that the probability of exactly one occurrence of King and Jack adjacent is 0.37193.

The proposer remarked that the problem occurs (incorrectly solved) in the volume *The World's Best Book of Magic* by Walter B. Gibson. The author assures the amateur prestidigitator that the required probability is about 2/3, on the reasoning that there are usually eight cards adjacent to the Kings and that each of these has one chance in twelve of being a Jack; $8(1/12) = 2/3$.

A Property of the Astroid

E 731 [1946, 394]. *Proposed by Howard Grossman, New York City*

At any point P on the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$, the tangent to the curve is the shortest line through P lying between the axes.

Solution by C. F. Pinzka, North Plainfield, N. J. The pencil of lines passing through $P(x_1, y_1)$ is given by $y-y_1=m(x-x_1)$, the intercepts being $x_0=x_1-y_1/m$ and $y_0=y_1-mx_1$. Then the length of any line through P is $(x_0^2+y_0^2)^{1/2}$. Applying the usual criteria we find that for the minimum line we have $m=-(y_1/x_1)^{1/3}$. But the slope of the tangent at P to the hypocycloid is found, upon differentiation, also to be $-(y_1/x_1)^{1/3}$. Hence these two lines are coincident.

Also solved by Michael Aissen, D. W. Alling, Murray Barbour, P. T. Bateman, Paul Brock, H. E. Fettis, I. M. Gardoff, E. A. Jacobs, H. L. Lee, S. T. Parker, C. R. Perisho, D. W. Matlock, W. R. McEwen, Emanuel Mehr, Herbert Reisman, E. D. Schell, W. R. Talbot, and P. D. Thomas.

A Three Digit Number

E 732 [1946, 394]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find, in the system of base 9, a number of three digits which, when transformed to the system of base 13, is composed of the same three digits.

Solution by Walter Penney, Washington, D. C. Let abc be the required number expressed in base 9. Since the leading digit cannot be the same in the two representations we have the following four cases:

$$81a + 9b + c = 169b + 13a + c,$$
$$81a + 9b + c = 169b + 13c + a,$$
$$81a + 9b + c = 169c + 13a + b,$$
$$81a + 9b + c = 169c + 13b + a.$$

The first equation, which reduces to $40b=17a$, obviously has no solution in integers less than 9. The second equation, which reduces to $40b=20a-3c$, yields the four solutions $abc=210$, 420, 630, 840. The third equation, which reduces to $42c=17a+2b$, yields the two solutions $abc=241$, 482. The last equation is found to yield no solution.

There are therefore the six solutions 210, 420, 630, 840, 241, and 482, expressed in base 9, corresponding to 102, 204, 306, 408, 124, and 248, respectively, in base 13.

Also solved by Michael Aissen, D. W. Alling, Murray Barbour, R. G. Blake, W. E. Bunyan, Monte Dernham, H. L. Lee, B. R. Leeds, Margaret Olmsted, S. T. Parker, Clay Perry, C. F. Pinzka, F. W. Saunders, E. D. Schell, W. R. Talbot, and the proposer. Many of these solutions gave only partial results.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4239. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

$AXBZ$ is a jointed rhombus connected with a fixed point O by two equal rods OA, OB. $OCZD$ is a jointed rhombus and YC, YD are equal rods. (Two Peaucellier cells, as it were "cross joined.") Prove that, as Y describes a circle, X describes a conic.

4240. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the relations which must connect N, B, B' in order that the number N may be written with the same three digits in the system of numeration of base B as in the system of base B'. Having given B, find B' and N. Apply the results when $B=10$.

4241. *Proposed by R. Goormaghtigh, Bruges, Belgium*

Consider a parabola having its vertex at a variable point M on a given plane curve, and its focus at F, the point dividing the radius of curvature MC in a constant ratio; the parabola touches its complete envelope at M and also at two other finite points. The corresponding chord of contact is perpendicular to the line joining M to the midpoint of the radius of curvature at C of the evolute of the given curve. If this chord of contact intersects MF at D, then $DC=2MF$.

4242. *Proposed by W. O. Pennell, Exeter, New Hampshire*

Determine the sums of the following infinite series:

$$(1)\qquad \frac{n+2}{n+1}-\frac{2n+2}{(n+1)(2n+1)}+\frac{3n+2}{(n+1)(2n+1)(3n+1)}-\cdots,$$

$$(2)\qquad \frac{3n+7}{n+1}-\frac{8n+9}{(n+1)(2n+1)}+\frac{15n+11}{(n+1)(2n+1)(3n+1)}-\frac{24n+13}{(n+1)(2n+1)(3n+1)(4n+1)}+\cdots,$$

where n is any real number except 0, -1, $-1/2$, $-1/3$, etc.

4243. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The point M, situated in the interior of a tetrahedron $ABCD$, such that the volume of the tetrahedron having for vertices the points of intersection of the

lines AM, BM, CM, DM with the opposite faces of $ABCD$ be a maximum, coincides with the centroid of $ABCD$.

SOLUTIONS

Convex Polyhedra

4176 [1945, 522]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers.

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

Solution by Leo Moser, University of Toronto. We prove (b) first. Any polygon having central symmetry is necessarily a parallel-sided $2m$-gon, since the point of central symmetry leads from any side to an equal and parallel side. We dissect each face into parallelograms. This can be done since every parallel-sided $2m$-gon can be dissected into a parallel sided $2(m-1)$-gon and a ribbon of $m-1$ parallelograms. Each edge of the polyhedron then determines a set of parallel sides and so a set of parallelograms called a zone. Now every parallelogram determines two zones which, by topological considerations, must intersect in another parallelogram which we will call the mate of the first parallelogram.

A parallelogram and its mate are congruent and parallel since the sides of the parallelograms are equal and parallel in pairs. Further a parallelogram cannot be parallel to more than one parallelogram, for then the polyhedron would be concave.

Clearly, two congruent parallelograms having sides parallel in pairs have a center of symmetry so that it only remains to show that the center of symmetry determined by one parallelogram and its mate is the same as that determined by any other parallelogram and its mate. We show this first for any two adjoining parallelograms and their mates. The center of symmetry for any parallelogram and its mate is determined by any point on one of the parallelograms. Take the point to be a point on a side of the parallelogram. Since this point belongs also to the adjoining parallelogram these two parallelograms and their mates must have the same center of symmetry. But we can pass from any parallelogram to any other parallelogram by crossing over a number of faces and edges. Hence all pairs of parallelograms have the same center of symmetry and the theorem is proved.

The proof of (a) follows easily. Every two zones (see b) must intersect. For, consider a fixed zone and a parallelogram not in it. Then the mate of this parallelogram is on the opposite side of the zone. Otherwise the polyhedron would be concave. Hence every zone intersects the zones determined by any parallelogram not in it, and so intersects every other zone. Further, as in (b), once two zones intersect they must intersect twice. Let the number of zones be n. Then the number of faces must equal the number of intersections of zones which is $2\binom{n}{2}=n(n-1)$ as required.

Probability of Two Coincident Birthdays

4177 [1945, 522]. *Proposed by P. R. Halmos, Syracuse University*

What is the smallest number of people sufficient to ensure that the probability that there be at least two with the same birthday is at least 1/2? It is to be assumed that any two days of the year are equally likely to be the birthday of a given individual and that there are no leap years.

Solution by Z. I. Mosesson, Fort Monroe, Va.

The probability that no two of n people have the same birthday is

$$\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\cdots\left(\frac{366-n}{365}\right)=\frac{1}{(365)^n}\,\frac{365!}{(365-n)!}.$$

Hence the probability that at least two of n people have the same birthday is

$$1-\frac{365!}{(365)^n(365-n)!}.$$

We wish to determine the least value of n for which

$$1-\frac{365!}{(365)^n(365-n)!}\geqq\frac{1}{2}\quad\text{or}\quad\frac{(365)^n(365-n)!}{365!}\geqq 2$$

or

$$n\log_{10}365+\log_{10}(365-n)!\geqq\log_{10}2+\log_{10}365!.$$

Using a table of $\log_{10}n!$ such as is given in T. C. Fry's *Probability and its Engineering Uses*, it is easy to show by trial and error that $n=23$.

Solved also by David Alling, P. T. Bateman, D. H. Browne, H. S. M. Coxeter, Monte Dernham, Howard Eves, Bart Park and the proposer.

Editorial Note. Browne, Coxeter, and Eves stated that the problem and its solution are given in Ball-Coxeter's *Mathematical Recreations and Essays*, p. 45.

A Maximum Product

4180 [1945, 523]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The product of k positive integers whose sum is N, where $N=kp+h$, is a maximum when h of the factors are equal to $p+1$ and the $k-h$ others to p.

Dedicated to E. P. Starke.

Solution by P. T. Bateman, Bryn Mawr College. It is evidently intended that $0\leqq h<k$. If the difference between the largest and the smallest of a set of k integers is unity, then the set clearly consists of h elements $p+1$ and $k-h$ elements p. If the difference between the largest and the smallest integer of the set is greater than unity, we can construct a new set having the same number of integers, and the same sum, but a greater product. This is done by replacing one of the maximal elements M by $M-1$ and one of the minimal elements m by

$m+1$. Since $M-m>1$,

$$(M-1)(m+1) = Mm + M - m - 1 > Mm$$

and thus the product is increased. Since the number of possibilities is finite, the stated result follows.

Solved also by Murray Barbour, B. P. Gill, H. S. Grant, P. C. Hammer, and P. A. Piza.

Editorial Note. Similar elementary reasoning shows that, k being unspecified, the maximum product which can be formed from any partition of N ($N>1$) into positive integers is given by

$$3^{N/3}, \qquad 2^2 \cdot 3^{(N-4)/3}, \qquad 2 \cdot 3^{(N-2)/3},$$

according as N is congruent to 0, 1, 2, respectively, modulo 3. For any positive integer q we can write

$$q \leqq 2^{q/2}, \quad \text{or} \quad q \leqq 3 \cdot 2^{(q-3)/2}, \tag{1}$$

according as q is even or odd. By the original result, the maximum product for any k is in the form $p^r(p+1)^s$. Without decreasing this product and without altering the sum of the factors we can replace it, according to (1), by a product of 2's and 3's. Finally, whenever the exponent of 2 exceeds 2, we can increase the product further by replacing 2^3 by 3^2 as often as possible. No further increase is now possible and the result appears as stated. The reader might care to go further and determine the maximum product of factors whose sum is N, when neither the factors nor the sum need be integers.

Generalized Simson Lines

4181 [1945, 582]. *Proposed by P. D. Thomas, Lumberton, Miss.*

Lines are drawn from a point P on the circumcircle of an equilateral triangle parallel to the three sides, thus determining six points, two on each side respectively. (1) Prove that the six points thus determined lie by threes on two straight lines. (2) If Q is the point of intersection of these two lines, find the locus of Q as P moves on the circumcircle.

Solution by R. Goormaghtigh, Bruges, Belgium. Consider a point P on the circumcircle Γ of any triangle $A_1A_2A_3$. In a system of complex coördinates having Γ as base circle (radius unity, center at origin), let t_1, t_2, t_3, and τ be the coördinates of A_1, A_2, A_3, and P, respectively, and let

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

If $\bar{a}$ is the conjugate of a, then $t_i\bar{t}_i = \tau\bar{\tau} = 1$ because Γ is the base circle. If $\lambda = e^{2\omega}$, it is easily verified that the side A_2A_3 and the line drawn from P forming with A_2A_3 the angle θ, will have the respective equations:

$$z + t_2t_3\bar{z} = t_2 + t_3, \qquad z + \lambda t_2t_3\bar{z} = \tau + \lambda t_2t_3\bar{\tau}.$$

Their intersection is given by

$$z_1 = \frac{\lambda t_2 + \lambda t_3 - \lambda t_2 t_3 \bar{\tau} - \tau}{\lambda - 1}.$$

The line

$$\tau z + \lambda s_3 \bar{z} = \frac{\tau^2 - \lambda s_1 \tau + \lambda s_2 - \lambda^2 s_3 \bar{\tau}}{1 - \lambda} \tag{1}$$

is easily seen to be satisfied by z_1, and also by the analogous points, z_2 and z_3, on the other sides of the triangle. This line is called the Simson line, for the angle θ, of P as to the given triangle. Since the two straight lines considered in the problem are obviously the Simson lines, for the angles $\pm\pi/3$, of P as to the given equilateral triangle, the proof of the first part is complete.

The Simson line, for a second angle θ', of P is given by (1) with λ replaced by $\lambda' = e^{2i\theta'}$. The intersection $Q(x)$ of these two lines is given by

$$x = \frac{\lambda\lambda'}{(1-\lambda)(1-\lambda')}\left[s_3\bar{\tau}^2 - s_2\bar{\tau} + s_1 - \frac{(\lambda + \lambda' - 1)\tau}{\lambda\lambda'}\right]. \tag{2}$$

Hence the locus of Q is in general a quartic. When the triangle $A_1A_2A_3$ is equilateral, so that $s_1 = s_2 = 0$, (2) reduces to a trinodal hypotrochoid. If also, as in the present problem, $\theta = -\theta' = \pi/3$, (2) becomes the required locus

$$x = \frac{s_3\bar{\tau}^2 + 2\tau}{3}, \tag{3}$$

which is the deltoid (three-cusped hypocycloid) having its cusps at the vertices of the given triangle.

Further results of interest, related to an equilateral triangle, may be obtained from (2). The locus of intersections of Simson lines, for the angles $\pm\pi/6$, is the circumcircle Γ. For the angles $\pm\pi/4$, (2) gives the regular trifolium. If we hold P fixed while the triangle turns about its center, the locus of the intersection of the Simson lines for the angles $\pm\theta$ is a circle passing through P.

Solved also by H. E. Fettis, Ou Li, Irma Moses, and O. J. Ramler, using analysis similar to the above (and with references to Morley and Morley, *Inversive Geometry*); by J. H. Butchart, Howard Eves, and Richard Meyer, using synthetic geometry; by Claire F. Adler, W. E. Cox, D. H. Erkiletian, Jr., W. A. Rees, W. T. Short, A. Sisk, G. A. Williams, and R. H. Wilson, Jr., using standard analytic geometry; and by W. J. Robinson, using both the latter methods.

Envelope of Simson Lines

4115 [1944, 233]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland*

From a point P on the circumcircle of a triangle lines are drawn inclined at angles θ to the sides of the triangle and meeting them in three collinear points.

Prove that as P varies the line on the three points envelopes a three-cusped hypocycloid. Prove that this hypocycloid is the locus of a point on a circle of radius $R/2 \sin\theta$ which rolls inside another circle three times the radius, whose center X is equidistant from the circumcenter C and the orthocenter O and is such that angle $OXC=2\theta$, R being the radius of the circumcircle.

Solution by the Proposer. Take the circumcircle as the unit circle with center at the origin, denote the vertices by the complex constants, t_1, t_2, t_3, and set

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3.$$

If τ denotes the point P, the equation of this generalized Simson line is

$$\bar{z}(1-\lambda)s_3\tau - z(1-\bar{\lambda})\tau^2 = \tau^3\bar{\lambda} - s_1\tau^2 + s_2\tau - s_3\lambda, \tag{1}$$

where $\lambda=e^{2i\theta}$. Consider also the circle

$$z = \frac{s_1}{1-\bar{\lambda}} + \frac{u}{2\sin\theta} \tag{2}$$

where $u\bar{u}=1$. (1) and (2) intersect where

$$u = i\tau e^{-i\theta}, \qquad \frac{s_3e^{3i\theta}}{i\tau^2}, \tag{3}$$

that is, u and $-s_3e^{i\theta}/iu^2$.

If $\theta=\pi/2$, the line (1) is the ordinary Simson line, the circle (2) is the nine-point circle, and the points (3) are u, $-s_3/u^2$. Therefore, the envelope when $\theta\neq\pi/2$ may be derived from the familiar result when $\theta=\pi/2$ by merely replacing the Simson line by the line (1), the nine-point circle by the circle (2), and s_3 by $s_3e^{i\theta}/i$. This gives the required results—and the additional information that the tangents at the cusps when $\theta\neq\pi/2$ are inclined to those in the case $\theta=\pi/2$ at angles $(\theta-\pi/2+2n\pi)/3$.

Editorial Note. In connection with his solution of 4181, O. J. Ramler points out that the envelope of the Simson line, for the angle θ, of P (see (1) in the solution of 4181 above) is obtained by differentiating partially with respect to τ. This gives

$$z + \frac{\lambda^2 s_1}{1-\lambda} = s_3\bar{\tau}^2 + 2\tau,$$

which is a deltoid similar to (3) in the solution of 4181, but three times as large. In the case of an equilateral triangle $A_1A_2A_3$, $s_1=0$ and the two deltoids are concentric.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

College Algebra. By A. A. Albert. New York and London, McGraw-Hill Book Co., 1946. 12+278 pages. $2.75.

In his introduction the author justifies his new textbook in college algebra and one can do no better than to let him present his own case: "College algebra has been a most abused subject. The time allotted to it is frequently inadequate for a genuinely good treatment, and indeed the entire course is sometimes omitted. This is due partly to a desire to bring students to a study of the calculus as early as possible. It is also due partly to the presentation of college algebra, in all texts thus far published, as a collection of seemingly unrelated topics. The desire to teach the calculus as early as possible tends to defeat its own ends. The building of a course in the calculus on what must be a weak foundation cannot result in a good student understanding of the subject. There is also no reason why the material of college algebra cannot be cohesively organized."

There is considerable justification for these views. Many college courses and texts have in the main persisted as collections of trick questions with too little emphasis on their systematic background. An excellent example of this type of presentation could until quite recently be found in the preparations for the first actuarial examinations. In algebra as in other fields of college instruction in mathematics there also exists a considerable amount of inertia which tends to make the choice of content in many courses almost dogmatically fixed. There are, of course, certain basic facts which will be with us forever, and which always must be included in the elementary books. But on the whole the newer trends in research and in the applications seem to be slow in exerting their influence. In mathematics there is no revolution as by quantum mechanics in physics, no atom bombs whose explosion immediately makes their impact felt down to the introductory texts.

The need for a revision of the instruction in elementary algebra, both in regard to content and presentation, fortunately has been recognized in a few of the recent texts and the present book is a significant step in the same direction.

The first few chapters give some of the traditional investigations of numbers, the laws of operation, the main properties of integers and the extension of the number system to the fields of real and complex numbers. Here in the introduction of the real numbers the reviewer finds one of the few instances where it seems that the presentation could be improved. The discussion is very brief and it may not give the students sufficient time and space to dwell on certain of the essential points.

In the subsequent chapters one finds the basic properties of polynomials up to unique factorization, the binomial theorem, and the method of indeterminate

coefficients. Two long chapters are devoted to the study of the algebraic equations. One finds, to begin with, a discussion of the decomposition into linear factors and of the relations between roots and coefficients. Imaginary roots and multiple roots are touched upon and the determination of rational roots is treated at some length. The determination of the real roots is discussed rather exhaustively, including bounds for the roots, methods for isolating roots, Descartes' rule of signs, Sturm's theorem, Horner's and Newton's methods for the computation of the roots. A useful feature is the inclusion of a chapter on vectors with applications to complex numbers, de Moivre's theorem, rotation of axes and conic sections.

The two last chapters represent an innovation for textbooks of this kind, containing a general discussion of determinants and matrix theory. This seems a most desirable inclusion. During the last decennials these topics have acquired an importance which is rapidly growing, not only in pure mathematics, but also in modern physical theories as well as in some of the now most common statistical methods. It is appropriate therefore that they should at present be given more generally and at an earlier level than previously. The author discusses determinants, their expansions and basic properties, and of course, their use in solving systems of linear equations. The matrix theory uses vectors in n-dimensional space, it includes multiplication of matrices, computation of inverses, elementary transformations and rank together with some discussion of similarity, symmetric and orthogonal matrices and applications to quadratic forms. Although it is clear that these themes cannot be given in their most general and complete form, there is enough to give the students some familiarity with the concepts.

To sum up, Albert's new textbook should prove a very valuable introduction to college algebra, both through its systematic outlook and through its inclusion of modern topics. The numerous problems add to its usefulness.

OYSTEIN ORE

Curves. By Lt. Col. R. C. Yates, AUS. West Point, N. Y., Department of Mathematics, United States Military Academy, 1946. 8+230 pages.

The detailed study of special plane curves is one of the most beautiful and at the same time one of the most neglected portions of the undergraduate curriculum. Its study is made more difficult by the fact that information concerning these curves is scattered widely throughout the mathematical literature, frequently in sources not easily obtainable in this country. And so it is a major operation to determine what are the known facts about many of the less common curves.

Colonel Yates has gone a long way toward remedying this difficulty by assembling the material contained in this book. Since there is no general theory of curves, the material is arranged in self-contained sections, one for each type of curve arranged alphabetically by the name of the curve. In each of these sections the reader will find a sketch of the history of the curve, a general description of

the curve including a graph, and its equations in a number of forms such as (1) cartesian, (2) polar, (3) parametric, (4) pedal, (5) Whewell, and (6) Cesaro forms. Further there is a collection of metric formulae for arc length, area, curvature, and the like, followed by a discussion of its relations to other curves and other general properties; and finally there is a list of references to the literature. There are also sections describing general methods of generating curves such as evolutes, envelopes, parallel curves, and pursuit curves, and there is a full section of methods of curve sketching. A table of contents and a detailed index provide ready access to this material.

The book is printed by offset from typescript and has a spiral binding. According to the title page it is "Prepared for use in the Department of Mathematics U. S. Military Academy," and it gives the impression of being a set of notes on curves rather than a textbook or a comprehensive treatise. In particular its shows a need for careful editing for it contains certain items which would not bother a well-trained reader, but which would confuse undergraduates. For example the first page contains a list of notation, presumably standard for the entire book. In this list p is defined as the "Distance from Origin to Tangent," but p appears in many other places with different meanings. Again on page 41 there is the statement: "for the conic, a tangent may be defined as a line meeting the curve in but one point" (!). The references suffer from not being in standard bibliographical form, and contain abbreviations and omissions of relevant data which complicate the task of locating the given material. It would be an injustice to the author, however, to continue such a list; for he has performed a very useful task in collecting such a wealth of facts on curves and has produced a book to which many mathematicians will turn with pleasure and profit.

C. B. ALLENDOERFER

Calculus. Second Edition. By F. H. Miller. New York, John Wiley and Sons, 1946, 16+416 pages. $3.50.

Those familiar with the author's first edition of *Calculus*, 1939, will welcome this new edition. Many changes have been made. The treatment of the limit concept has been amplified. An article on graphical differentiation and one on approximation integration will contribute to the usefulness of the book. The exercises have been revised and some of the more complicated problems in the first edition have been replaced.

Throughout Chapters II and III the derivative of y with respect to x is denoted by $D_x y$, and the ordinary notation for the derivative is not used until after the introduction of differentials. The natural logarithm of x is denoted by $\ln x$, and the common logarithm by $\log x$. These changes seem to be in keeping with current trends. In dealing with indeterminate forms the symbols $0/0$, ∞/∞, and so on, are avoided.

The author has written a detailed and carefully planned textbook which is directed toward the needs and interests of not only the future major in mathematics, but also the engineer. Geometrical and physical interpretations of both

differentiation and integration are introduced before the general processes of these subjects are discussed. Thus the student is given an immediate idea of the usefulness of the calculus before he is exposed to long lists of formulas.

Many readers will enjoy the manner in which Duhamel's theorem is avoided in the formulation of geometrical and physical problems. In fact, infinitesimals of higher order are not mentioned, nor does the need for this concept arise in the book.

Throughout the book definitions and theorems are carefully worded and well displayed on the page. The chapter on limits is very thorough and the fundamental theorems on limits are proved in an interesting and fairly rigorous manner. A figure or two would have helped to illustrate the general definition of a limit of a sequence. Moreover, the transition from the limit of a sequence to the limit of a function is a little abrupt. There is a good discussion of continuity, and throughout the book the author is careful to state what assumptions as to continuity are assumed and used. The average student, however, will find this chapter hard reading and much of the material will have to be broken down by the instructor.

Chapter IX on partial differentiation comes before the chapters on integration. Here the addition of figures illustrating functions of two variables and showing the geometrical meaning of the partial derivatives would have improved the discussion. There is no mention of directional derivatives.

In Chapter XI the definite integral is first presented as a limit of a certain sum and is shown to be a function of its upper limit. Then it is shown analytically to possess a derivative and hence the concept of an integral as an antiderivative arises. In this way the student is taught at the outset that the definite integral need not necessarily mean an area. An example actually showing the calculation of a limit of a sum would have been a welcome addition to the chapter.

There is a chapter on infinite series which includes Maclaurin's Integral Test and Taylor's Series with a remainder. The discussion of operations with power series is brief but to the point and the student is given ample warning about differentiating and integrating series.

In the chapter on multiple integrals the author is careful to distinguish between double integrals and iterated integrals. The applications mentioned here include most of the standard ones. There is a good discussion of the area of a surface.

The book ends with a chapter on differential equations. Here the discussion of separation of variables could be more rigorous. Moreover the solution of differential equations in series expansions is not included.

A few matters of detail might be mentioned. Exercises 14 and 15, p. 32, would be better placed after motion in the plane curve on p. 125. There are no problems to go with the discussion of derivatives of parametric equations on p. 42.

Some teachers will object to the introduction of the term infinitesimal on

page 7. Many modern texts are avoiding the term altogether. Also there will be disagreement with the discussion of differentials as presented, especially over the use of the equation $dy=f'(x)\Delta x$, with $f(x)=x$, to justify the taking of dx to be always equal to Δx. It is a mistake to give students the impression that dx is always equal to Δx.

On page 66 hyperbolic functions are introduced as exercises and there is no discussion of these functions in the text. In Exercise 43 on that page the student is asked to show that $\sinh^{-1}y=\ln(y+\sqrt{1+y^2})$. It is the reviewer's experience that students need a text discussion as well as help from the instructor in order to work such problems with confidence.

The discussion of maxima and minima does not include examples in which the critical point has an infinite slope. Nor are there problems in which the absolute maximum in a certain range occurs at a point for which $f'(x)\neq 0$.

The reviewer found few misprints and none of a serious nature. In places the text material looks a bit crowded and some of the discussions are too brief but considering the amount of material covered the author has written a clear and useful text and one in which applications are certainly not lacking. Answers are given to the odd-numbered exercises and those to the even-numbered ones are available. An appendix of algebra, geometry, trigonometry, and analytic geometry formulas is included. There are tables of the functions e^x, e^{-x}, $\ln x$, $\log x$, as well as a fairly complete table of integrals. A table of differentiation formulas is needed, for nowhere are such formulas listed for easy reference. All the excellent features of the original edition have been retained. The changes have added desirable features and should increase the effectiveness of the text as a teaching instrument.

C. D. Olds

NEW BOOKS RECEIVED

Analytic Geometry and Calculus. By J. F. Randolph and Mark Kac. New York, Macmillan Co., 1946. 9+642 pages. $4.75.

Basic Mathematics for Technical Courses. By C. E. Tuites. New York, Prentice-Hall Co., 1946. 14+344+132 pages. $5.00.

Mathematical Aids for Engineers. By R. W. Dull. New York and London, McGraw-Hill Book Co., 1946. 12+346 pages. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

Of probable interest to members of mathematical clubs is the series of articles appearing in *The Pentagon*, official publication of *Kappa Mu Epsilon*, presenting bibliographies on subjects which are suitable for club programs. The bibliographies thus far presented together with the issue in which they appear are:

Women as mathematicians, vol. V, no. 2, Spring 1946
The cattle problem of Archimedes, vol. V, no. 2, Spring 1946
Paper folding, vol. V, no. 2, Spring 1946
Mathematical prodigies, vol. VI, no. 1, Fall 1946
Calculating machines, vol. VI, no. 1, Fall 1946
The bee as a mathematician, vol. VI, no. 1, Fall 1946

Some of the mathematics clubs have sent to the editor of this department their programs announcing coming events. Many of these programs have been very clever and stimulating. The editor is of the opinion that unusual effort in the advertising of events to students is well worth while in creating interest. Miss Marion Stark, Faculty Advisor of the Wellesley College *Mathematics Club*, contributes the following, which was mimeographed on a suitable folder:

"The Wellesley College Mathematics Club, a member of the Intercollegiate Mathematics Clubs of Greater Boston, cordially welcomes you to join in this year's activities.

"Our purpose is to take *Mathematics* off the bookshelves, dust it off, and have fun with it. *Mathematics* has been scaring us long enough; we feel it's time we put it in its place—and even play a few tricks on it. (Of course the joke may be on us, but that's all part of the fun.) The members of the club include everybody taking second and third grade *Mathematics*, and all juniors and seniors taking a first grade course.

"We'll keep you posted as to when and where the meetings will take place. We hope you'll come and help us make merry with deltas and epsilons. Remember—no dues, no duties, no drudgery; just fun, food, and entertainment, and (who knows?) you might even meet a handsome Tech student!"

CLUB REPORTS, 1945-46

Delta-Y Club, D'Youville College

As an increment to the courses in mathematics, the *Delta-Y Club* took as a general topic for study, *History of Mathematics*. At regular monthly meetings there were prepared papers and informal discussions on:

Primitive mathematics by Marjorie Benzinger
Mathematics takes wings by Jane Deckop
René Descartes by Alice Staebell

Apollonius of Perga by Evelyn Kruse
The problem of Apollonius by Mary Loretta Hoar
Leibniz by Helen J. Hand

The activities for the year ended in May with a social meeting and spread. Guest speaker was Mrs. Louis D. Copley.

Officers for 1945–46: President, Mary Loretta Hoar; Secretary-Treasurer, Helen J. Rand.

Kappa Mu Epsilon, Hofstra College

The following papers were presented at regular meetings during the year:
Mathematical problems of antiquity, by Ruth Mayer
Atomic energy, by Wanda Scala
Mathematics in chemistry, by Professor J. George Lutz
Problems of mechanics, by Professor A. D. Capuro
Mathematics and physics of the nerves, by Professor Otto Schmidt, formerly of the University of Minnesota, with demonstration of the lie detector in action.

In April, a joint supper meeting was held at Adelphi College with the *Adelphi Mathematics Club*. At this meeting the following talks were given:
Conic sections, illustrated with light projections, lantern slides, and string models, by Professor H. von Baravalle of Adelphi.
Kappa Mu Epsilon, by Professor L. F. Ollmann
The mathematical theory of spring fever, by Professor E. R. Stabler.

At the annual initiation banquet the main address of the evening was:
A comparison of American and English colleges, by Mr. E. Trudeau Thomas, Hofstra director of admissions.

Other activities of the year included a Christmas square dance party, and a picnic in June at Belmont Lake State Park.

Officers for the year 1946–47 are: President, Edward Ryder; Vice-President and Secretary, Leo Malone; Treasurer, Professor J. George Lutz; Corresponding Secretary, Professor Albert Capuro; Faculty Sponsor, Professor L. F. Ollmann.

Mathematics Club, Wellesley College

The club assisted the Department of Mathematics in the presentation of two exhibitions. The first was a model exhibition for which the club had a room of mathematical toys, games and puzzles. The second, an exhibition of rare books in the College Library, at which the club took charge of an international table, containing books by mathematicians from as many countries as possible.

Students participated in two club programs by presenting talks on *Mathematics used in summer jobs* which they had held, and *Famous mathematicians of other countries*, as part of a United Nations Program at the College. At other meetings the members sang songs composed by students and faculty.

Early days in Wellesley's mathematics was the subject of a talk by Miss Helen Merrill, Emeritus Professor of Mathematics. Professor W. R. Ransom gave a lecture to the club at another meeting. Appreciation of the service rendered by Miss Jennie Copeland, retiring chairman of the mathematics depart-

ment was expressed in the form of a supper meeting at the close of the year at which time a gift was presented to her.

The officers for 1945–46 were: President, Ida Harrison; Vice-President, Eileen McGuire; Treasurer, Jean Marshall; Secretary, Patricia Peare; Junior Executive, Lois Wood; Faculty Advisor, Miss Marion Stark.

Kappa Mu Epsilon, Texas Technological College

Seven meetings were held during the year, including two formal initiation programs and a Christmas party. Eleven new members were initiated on November 23, 1945 and fourteen on May 3, 1946.

Among the several interesting papers presented during the year was one on: *A mathematical theory of aeronautics* by Assistant Professor Lida B. May.

The officers for the year 1946–47 are: President, James Fergerson; Vice-President, Helen Robin; Secretary, Betty Jones; Treasurer, Joel Simmons; Faculty Sponsor, Miss Virginia Bowman; Secretary Descartes, Miss Lida B. May.

Kappa Mu Epsilon, University of New Mexico

Monthly meetings were held by the *New Mexico Alpha* Chapter of *Kappa Mu Epsilon* with an initiation each term. The main event of the year was the visit of Dr. Fraenkel of the University of Jerusalem to the campus. At an open meeting sponsored by the chapter, Dr. Fraenkel spoke on *The recent controversies about the foundation of mathematics.*

Programs for the year included the following talks:

Tesselations, by Mrs. Eupha Buck Morris
Photogrammetry, by Mrs. Ruth Kendrick
Our calendar, by Dr. Martin Fleck
The Central Valley project, by Mr. Marvin May
Functions of the polar planimeter, by Dr. Arthur Rosenthal
How to win on the horses, by Dr. H. D. Larsen
Magic squares, by Mr. Frank Lane.

The officers for 1945–46 were: President, Bob Fox; Vice-President, William E. Dickerson; Secretary, Ted Hawley; Treasurer, Merle Mitchell; Faculty Sponsor, Miss E. Marie Hove.

The following officers were elected for 1946–47: President, Darrell Baker; Vice-President, Paul Barnhart; Secretary, Dorothy Lodter; Faculty Sponsor, Mrs. Eupha Buck Morris.

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

THE SEVENTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The seventh annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, May 24, 1947. This Competition, made possible by the Trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor George Mackey, Hunt Hall 12, Harvard University, Cambridge, Massachusetts, by a postcard request. All applications must be filed with Professor Mackey not later than May 1, 1947. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination, the books are to be sent by registered mail or by express to Professor Mackey, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$400, \$300, \$200 and \$100 in the order of their rank. In addition, there will be prizes of \$40, \$30, \$20 and \$10 awarded to the members of these teams according to the rank of the team; a prize of \$50 to each of the five highest contestants, and a prize of \$20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to the fifteen individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (at Radcliffe College, in the case of a woman) an annual \$1000 William Lowell Putnam Prize Scholarship to one of the first five contest-

ants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the six previous competitions and examination questions will be found in the MONTHLY for May 1938, 1939, 1940, 1941 and 1942, and for October, 1946.

BROWN UNIVERSITY

The establishment of a graduate department of the History of Mathematics at Brown University has been announced. This department will be under the leadership of Professor Otto Neugebauer. Associated with him will be Dr. A. J. Sachs, who has been promoted to an assistant professorship. A course in Oriental History, by Professor Sachs, is planned for the academic year 1947–1948 and it is expected that a seminar on selected topics in ancient astronomy and mathematics will be offered in cooperation with the departments of mathematics, astronomy and classics.

PRINCETON UNIVERSITY

A conference on the "Problems of Mathematics," one of a series of conferences in celebration of the Bicentennial of the founding of Princeton University, was held at Princeton on December 17, 18, and 19, 1946. The participants included, besides the local mathematicians, 76 from elsewhere, among whom 12 came from outside the United States. The conference was organized in the form of nine round tables on various subjects, with the discussion oriented as far as possible towards the formulation of problems for future work. The sessions were as follows: *Algebra*, chairman E. Artin, reporter G. P. Hochschild, discussion leaders G. Birkhoff, R. Brauer, N. Jacobson; *Algebraic Geometry*, chairman S. Lefschetz, reporter I. S. Cohen, discussion leaders W. V. D. Hodge, O. Zariski; *Differential Geometry*, chairman O. Veblen, reporter C. B. Allendoerfer, discussion leaders V. Hlavatý, T. Y. Thomas; *Mathematical Logic*, chairman A. Church, reporter J. C. C. McKinsey, discussion leader A. Tarski; *Topology*, chairman A. W. Tucker, reporter S. Eilenberg, discussion leaders H. Hopf, D. Montgomery, N. E. Steenrod, J. H. C. Whitehead; *New Fields*, chairman J. von Neumann, reporter V. Bargmann, discussion leaders G. C. Evans, F. D. Murnaghan, J. L. Synge, N. Wiener; *Mathematical Probability*, chairman S. S. Wilks, reporter J. W. Tukey, discussion leaders H. Cramér, J. L. Doob, W. Feller; *Analysis*, chairman S. Bochner, reporter R. P. Boas, discussion leaders L. V. Ahlfors, E. Hille, M. Riesz, A. Zygmund; *Analysis in the Large*, chairman M. Morse, reporter M. Shiffman, discussion leaders R. Courant, H. Hopf.

It is planned to issue shortly a descriptive pamphlet, and later a more complete monograph, covering the work of the conference. The monograph will contain an extensive list of the problems proposed.

THE UNIVERSITY OF NORTH CAROLINA

The University of North Carolina completed in July 1946 the organization

of the all-University Institute of Statistics. Gertrude M. Cox is Director of the Institute, with W. G. Cochran and Harold Hotelling as Associate Directors in charge respectively of the Department of Experimental Statistics at Raleigh and the Department of Mathematical Statistics at Chapel Hill.

The Department of Mathematical Statistics is concerned primarily with research in the theory of statistics and with graduate training for the Ph.D. degree of students who expect to teach or develop statistical theory. Courses offered by regular members of the Department during the present year, 1946–47, include Statistical Inference, Mathematical Economics, and Least Squares and Time Series by Professor Hotelling; Correlation and Multivariate Analysis by Associate Professor P. L. Hsu; Introductory and Advanced Probability by Associate Professor Herbert Robbins; and Sequential Analysis and Rank Order Statistics by Instructor Edward Paulson. M. S. Bartlett of Cambridge University, Visiting Professor for the first half of the academic year 1946–47, offered courses on Stochastic Processes and on Estimation and Testing Hypotheses. Professor Harald Cramér of the University of Stockholm gave a series of three lectures in December 1946 on the Theory and Application of Stochastic Processes. Associate Professor W. G. Madow, now in Brazil, will join the Department at the end of 1947.

Students entering the Department of Mathematical Statistics must know mathematics through advanced calculus. Additional mathematics, including theory of functions and matrix algebra is required before the completion of their training. It is a fundamental principle of the Institute that students must acquire both a thorough knowledge of the mathematics of statistics and experience in a specific field of application. The latter is provided at Raleigh where the large staff deals with many branches of applied statistics, including sampling survey techniques, analysis of economic data, industrial statistics, design of experiments, and crop forecasting. Various research and consultation projects are in progress at Raleigh in which advanced students have an opportunity to work under critical supervision.

PERSONAL ITEMS

Professor J. A. H. Duffie of Assumption College, University of Western Ontario has been appointed professor of chemistry at the University of Ottawa.

Professor J. P. Everett of Western Michigan College has retired with the title of professor emeritus.

J. E. Freund has been appointed to an assistant professorship at Alfred University, Alfred, New York.

Assistant Professor F. S. Harper of the University of Nebraska has been appointed to a professorship and head of the department of actuarial science at Drake University.

M. A. Hyman is employed as a mathematician at the Naval Ordnance Lab-

oratory in Washington, D. C. He is not at Rutgers University as was erroneously reported in the December issue of the MONTHLY.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Royal Danish Academy of Sciences and Letters and the Polish Academy of Sciences and Letters.

Professor Hermann Weyl of the Institute for Advanced Study has been elected a corresponding member of the Zürcher Naturforschende Gesellschaft and a foreign member of the Royal Swedish Academy of Sciences.

The following appointments to instructorships are announced:

Harvard University: Dr. R. C. James

Hofstra College: Frank Hawthorne

Queens College (Tutors in mathematics): Louise H. Ercolano, Florence E. Gerhardt, Aida Kalish, Mrs. Charlotte L. R. Knag, Bernice J. Lehrman

Northwestern University: Dr. A. O. Lindstrum

The University of Maine: J. A. Harmon, Leo Lapidus, Mrs. A. D. Mawhinney, Sutton Monro, Letitia Watson

The University of Illinois: Dr. Joseph Landin

Rider College: Emanuel Levine

D. E. Marrs of Pittsburg, California died May 28, 1946.

Dean Emeritus C. S. Slichter of the University of Wisconsin died October 4, 1946.

Associate Professor I. D. Stewart of Whitman College, Walla Walla, Washington, died July 21, 1946.

Professor G. B. Sweazey of Westminster College, Fulton, Missouri died August 10, 1946.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTIETH ANNUAL MEETING OF THE ASSOCIATION

The thirtieth annual meeting of the Mathematical Association of America was held at Swarthmore College, Swarthmore, Pennsylvania, on Thursday and Friday, December 26–27, 1946, in conjunction with meetings of the American Mathematical Society. About four hundred and ninety-eight persons attended the meetings, including the following two hundred and eighty members of the Association:

C. R. ADAMS, Brown University
LOUISE ADAMS, High Point College
V. W. ADKISSON, University of Arkansas
R. P. AGNEW, Cornell University
C. B. ALLENDOERFER, Haverford College
H. E. ARNOLD, Wesleyan University
EMIL ARTIN, Princeton University
SILVIO AURORA, Columbia University

FRANK AYRES, JR., Dickinson College
H. C. AYRES, U. S. Naval Academy
W. L. AYRES, Purdue University

R. P. BAILEY, U. S. Naval Academy
N. H. BALL, U. S. Naval Academy
D. H. BALLOU, Middlebury College
JOSHUA BARLAZ, Rutgers University
P. T. BATEMAN, Yale University
HELEN P. BEARD, Newcomb College
H. M. BEATTY, Ohio State University
A. A. BENNETT, Brown University
THEODORE BENNETT, Marietta College
A. L. BILLIG, High School, Allentown, Pa.
M. T. BIRD, Allegheny College
W. E. BLEICK, U. S. Naval Academy
R. P. BOAS, Brown University
T. A. BOTTS, University of Delaware
S. G. BOURNE, Johns Hopkins University
J. G. BOWKER, Middlebury College
H. W. BRINKMANN, Swarthmore College
FOSTER BROOKS, Kent State University
R. H. BRUCK, University of Wisconsin
C. T. BUMER, Bureau of Ordnance
R. S. BURINGTON, Bureau of Ordnance
HERBERT BUSEMANN, Smith College
J. H. BUSHEY, Hunter College
JEWELL HUGHES BUSHEY, Hunter College

S. S. CAIRNS, Syracuse University
ELEANOR CALKINS, College of William and Mary
E. J. CAMP, Macalester College
H. H. CAMPAIGNE, U. S. Navy
P. A. CARIS, University of Pennsylvania
W. B. CARVER, Cornell University
REV. J. E. CASE, St. Louis University
F. L. CELAURO, Lehigh University
J. O. CHELLEVOLD, Lehigh University
W. F. CHENEY, Jr., University of Connecticut
RANDOLPH CHURCH, U. S. Naval Academy
R. V. CHURCHILL, University of Michigan
J. A. CLARKSON, University of Pennsylvania
J. W. CLAWSON, Ursinus College
A. B. COBLE, University of Illinois
NATHANIEL COBURN, University of Michigan
NANCY COLE, Connecticut College
J. A. COOLEY, University of Tennessee
T. F. COPE, Queens College
RICHARD COURANT, New York University
W. H. H. COWLES, Pratt Institute
G. F. CRAMER, U. S. Navy
H. B. CURRY, Pennsylvania State College
J. H. CURTISS, National Bureau of Standards
E. H. CUTLER, Lehigh University

JAMES ELMER DAVIS, Drexel Institute of Technology
L. J. DECK, Muhlenberg College
F. F. DECKER, Syracuse University
CARL DENBOW, U. S. Naval Academy
F. L. DENNIS, Ursinus College
A. H. DIAMOND, Oklahoma A. and M. College
L. L. DINES, Smith College
H. L. DORWART, Washington and Jefferson College
ARNOLD DRESDEN, Swarthmore College
D. M. DRIBIN, War Department
ROY DUBISCH, Syracuse University
NELSON DUNFORD, Yale University
JANET C. DURAND, Vassar College

J. E. EATON, Naval Research Laboratory
E. D. EAVES, University of Tennessee
NAT EDMONSON, Johns Hopkins University
SAMUEL EILENBERG, Indiana University

W. H. FAGERSTROM, College of the City of New York
WILLIAM FELLER, Cornell University
F. A. FICKEN, University of Tennessee
N. J. FINE, Washington, D. C.
DANIEL FINKEL, Amherst College
L. R. FORD, Illinois Institute of Technology
TOMLINSON FORT, University of Georgia
R. M. FOSTER, Polytechnic Institute of Brooklyn
J. S. FRAME, Michigan State College

H. M. GEHMAN, University of Buffalo
B. H. GERE, U. S. Naval Academy
REV. F. J. GERST, Loyola University
J. H. GIESE, Ballistic Research Laboratories
B. P. GILL, College of the City of New York
J. W. GIVENS, Illinois Institute of Technology
A. M. GLEASON, Harvard University
MARY GOINS, Marshall College
MICHAEL GOLDBERG, Bureau of Ordnance
H. H. GOLDSTINE, Institute for Advanced Study
R. A. GOOD, University of Maryland
A. W. GOODMAN, Columbia University
S. H. GOULD, University of Toronto
A. A. GRAU, University of Kentucky
GEORGE GROSSMAN, DeWitt Clinton High School, New York
C. C. GROVE, University of Pennsylvania

V. H. HAAG, Hershey Junior College
THEODORE HAILPERIN, Lehigh University
R. W. HAMMING, Bell Telephone Laboratories
E. S. HAMMOND, Bowdoin College
C. E. HEILMAN, Syracuse University
M. H. HEINS, Brown University
G. C. HELME, Pratt Institute
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
R. P. HOBBS, Rinehart and Company, Inc.
D. L. HOLL, Iowa State College
T. R. HOLLCROFT, Wells College
D. B. HOUGHTON, Franklin Institute
E. MARIE HOVE, Hofstra College
M. GWENETH HUMPHREYS, Newcomb College
W. A. HURWITZ, Cornell University
H. D. HUSKEY, University of Oklahoma
W. R. HUTCHERSON, Berea College

J. E. IKENBERRY, Madison College
M. H. INGRAHAM, University of Wisconsin

E. D. JENKINS, Eastern Kentucky State Teachers College
FRITZ JOHN, New York University
ROBERTA F. JOHNSON, Wilson College
R. P. JOHNSON, Carnegie Institute of Technology
W. L. JOHNSON, Mississippi Southern College
B. W. JONES, Cornell University

MARK KAC, Cornell University
IRVING KAPLANSKY, University of Chicago
J. R. KLINE, University of Pennsylvania
H. L. KRALL, Pennsylvania State College
R. R. KUEBLER, JR., Dickinson College
H. W. KUHN, Ohio State University

MARY E. LADUE, Barnard College
GILLIE A. LAREW, Randolph-Macon Woman's College
J. A. LARRIVEE, University of Vermont
C. G. LATIMER, University of Kentucky
V. V. LATSHAW, Lehigh University
SOLOMON LEFSCHETZ, Princeton University
JOSEPH LEHNER, Hydrocarbon Research, New York
MARGUERITE LEHR, Bryn Mawr College
WALTER LEIGHTON, Washington University
CAROLINE A. LESTER, New York State College for Teachers
HARRY LEVY, University of Illinois
ANNE L. LEWIS, Woman's College, University of North Carolina
S. B. LITTAUER, Newark College of Engineering
A. T. LONSETH, Northwestern University

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
H. F. MAC NEISH, Brooklyn College
INGO MADDAUS, JR., Naval Research Laboratory
J. D. MANCILL, University of Alabama
F. L. MANNING, Ursinus College
R. H. MARQUIS, Ohio University
M. H. MARTIN, University of Maryland
W. T. MARTIN, Massachusetts Institute of Technology
E. D. MCCARTHY, University of Detroit
SOPHIA L. MCDONALD, University of California
D. L. MCDONOUGH, High School, Philadelphia
S. S. MCNEARY, Drexel Institute
MARY E. MEADE, University of Maryland
A. E. MEDER, JR., Rutgers University
EMANUEL MEHR, Johns Hopkins University
A. S. MERRILL, Montana State University
H. L. MEYER, JR., University of Chicago
A. N. MILGRAM, Institute for Advanced Study
D. D. MILLER, University of Tennessee
SOLOMON MITCHELL, Grinnell College
E. E. MOISE, University of Texas
MARTIN MOLIVER, Martin College
DEANE MONTGOMERY, Yale University
A. H. MOORE, Pratt Institute
LILLIAN MOORE, Far Rockaway High School, New York
RICHARD MORRIS, Rutgers University
IRMA R. MOSES, Temple University
E. D. MOUZON, JR., Southern Methodist University
C. W. MUNSHOWER, Colgate University
W. R. MURRAY, Franklin and Marshall College

C. A. NELSON, New Jersey College for Women
A. B. NEALE, Watson Laboratories
J. J. NEWMAN, Harvard University
C. V. NEWSOM, Oberlin College
P. B. NORMAN, Pratt Institute

C. O. OAKLEY, Haverford College
L. F. OLLMANN, Hofstra College
ISAAC OPATOWSKI, University of Michigan
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, Pennsylvania State College
J. C. OXTOBY, Bryn Mawr College

S. T. PARKER, University of Cincinnati
G. W. PATTERSON, University of Pennsylvania
E. K. PAXTON, Washington and Lee University
A. M. PEISER, Rutgers University
C. R. PERISHO, McCook Junior College
B. J. PETTIS, Yale University
H. R. PHALEN, College of William and Mary
C. R. PHELPS, Rutgers University
A. E. PITCHER, Lehigh University
G. B. PRICE, University of Kansas
A. L. PUTNAM, University of Chicago

J. F. RANDOLPH, Oberlin College
S. E. RASOR, Ohio State University
G. E. RAYNOR, Lehigh University
C. N. REYNOLDS, West Virginia University
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
R. F. RINEHART, Case School of Applied Science
R. M. ROBINSON, Princeton University
R. E. ROOT, U. S. Naval Academy
J. B. ROSENBACH, Carnegie Institute of Technology
LOUISE J. ROSENBAUM, Reed College
R. A. ROSENBAUM, Reed College
P. C. ROSENBLOOM, Syracuse University
J. B. ROSSER, Cornell University
L. J. ROUSE, University of Michigan

RAPHAEL SALEM, Massachusetts Institute of Technology
HANS SAMELSON, University of Michigan
R. G. SANGER, Kansas State College
ARTHUR SARD, Queens College
I. J. SCHOENBERG, University of Pennsylvania
WLADIMIR SEIDEL, University of Rochester
W. E. SEWELL, War Department, Army Education Branch
W. P. SHARP, JR., Newark College of Engineering
L. W. SHERIDAN, College of St. Thomas
D. T. SIGLEY, Johns Hopkins University
L. L. SMAIL, Lehigh University
C. V. L. SMITH, Raytheon Manufacturing Co.
W. M. SMITH, Lafayette College
ERNST SNAPPER, University of Southern California
ANDREW SOBCZYK, Watson Laboratories
F. W. SOHON, Georgetown University
P. I. SPEICHER, Albright College
E. R. STABLER, Hofstra College
R. C. STALEY, University of North Dakota
E. P. STARKE, Rutgers University
F. H. STEEN, Allegheny College
H. W. STEINHAUS, Equitable Life Assurance Society, New York
R. C. STEPHENS, Knox College
ELLEN C. STOKES, New York State College for Teachers
M. H. STONE, University of Chicago
HELEN F. STORY, Pennsylvania State College
IRVING SUSSMAN, University of Dayton
J. L. SYNGE, Carnegie Institute of Technology
OTTO SZÁSZ, University of Cincinnati
GABOR SZEGÖ, Stanford University

FEODOR THEILHEIMER, Trinity College
R. M. THRALL, University of Michigan
LEONARD TORNHEIM, University of Michigan
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University

G. L. WALKER, Temple University
R. J. WALKER, Cornell University
A. D. WALLACE, University of Pennsylvania
J. L. WALSH, Harvard University
JEAN B. WALTON, University of Pennsylvania
W. R. WASOW, Swarthmore College
C. W. WATKEYS, University of Rochester
G. C. WEBBER, University of Delaware
M. S. WEBSTER, Purdue University
J. V. WEHAUSEN, David Taylor Model Basin
MARIE J. WEISS, Newcomb College
E. T. WELMERS, Bell Aircraft Corporation
ANNA PELL WHEELER, Bryn Mawr College
J. H. WHITE, U. S. Naval Academy
A. L. WHITEMAN, Navy Department
E. A. WHITMAN, Carnegie Institute of Technology
P. M. WHITMAN, Tufts College
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, Texas Technological College
S. S. WILKS, Princeton University
K. P. WILLIAMS, Indiana University
MARY E. WILLIAMS, Skidmore College
W. L. WILLIAMS, University of South Carolina
R. H. WILSON, JR., Temple University
CLEMENT WINSTON, Department of Commerce
H. A. WOOD, Chance Vought Aircraft
FRANCES M. WRIGHT, Triple Cities College

BERTRAM YOOD, Yale University
J. W. T. YOUNGS, Indiana University

Rooms for members of the organizations and their families were provided in the college dormitories, and meals were served in the college dining room in Parrish Hall. Tea was served on Thursday and Friday afternoons in the parlor of Bond Memorial Hall by the ladies of the Swarthmore College Department of Mathematics. An excellent program of chamber music was presented Friday evening in the auditorium of the Clothier Memorial Building by Phyllis Groff, Leona Gold, Helen Arens, Betty Benthin, Robert N. Hilkert, Ralph H. Fox, and Arnold Dresden.

A dinner for the two organizations was held at 6:30 Friday evening in the college dining room. Professor C. O. Oakley acted as toast-master and introduced President John W. Nason of Swarthmore College who welcomed the visiting organizations in a few happily chosen words. He was followed by Professor W. T. Martin who spoke on the general subject of the post-war graduate student in mathematics. Resolutions were presented by Professor C. G. Latimer expressing the thanks of the visiting organizations to President Nason and the Board of Managers of Swarthmore College, to the members of the Swarthmore Mathematics Department and their wives, and to the various members of the College staff who helped with the friendly and efficient arrangements which made the meeting so enjoyable. The resolutions were adopted by a rising vote.

The American Mathematical Society held its sessions between noon Friday and noon Sunday. The Josiah Willard Gibbs lecture was delivered by Professor Subrahmanyan Chandrasekhar of the University of Chicago on Thursday evening, his subject being "The transfer of radiation in stellar atmospheres"; and on Saturday at 2:00 P.M. Professor A. P. Morse of the University of California gave an address, by invitation, on "Derivatives and their integrals."

The Mathematical Association held its sessions on Thursday afternoon and Friday morning, the program having been arranged by a committee consisting of I. J. Schoenberg, chairman, D. W. Hall, and Ivan Niven. The following papers were presented:

First Session of the Association

"Parallelism, solid angle and curvature," by Professor C. B. Allendoerfer, Haverford College.

"Mathematics in the Army Education Program," by Colonel W. E. Sewell, Army Education Branch.

"The theory of braids," by Professor Emil Artin, Princeton University.

Second Session of the Association

"On the multiplication of series," by Professor Antoni Zygmund, University of Pennsylvania.

"Random walk and the theory of Brownian motion," by Professor Mark Kac, Cornell University.

Professor M. S. Knebelman of the State College of Washington was to have presented a paper on "The teaching of college mathematics today" at the Friday morning session, but he was unfortunately unable to attend the meeting.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Thursday at 7:30 P.M. in the News Bureau room in Parrish Hall, seventeen members of the Board being present.

Among the more important items of business transacted were the following.

Ninety-two persons as listed below, were elected to membership on applications duly certified:

SISTER M. ANITA, A.M. (Seton Hall Coll.) Instr., Caldwell Coll., Caldwell, N. J.

D. L. ARENSON, Student, Illinois Inst. of Tech., Chicago, Ill.

R. P. BAILEY, Ph.D. (Pennsylvania) Asso. Prof., U. S. Naval Acad., Annapolis, Md.

DOROTHEA M. BAUMANN, B.S. (Marquette) Teacher, Rufus King High School, Milwaukee, Wis.

MAX BEBERMAN, B.S. (C.C.N.Y.) Teacher, Math. and Sci., Nome Territorial High School, Nome, Alaska

PAUL BELGODÈRE, Agrégé de Math. (Paris, E.N.S.) Secrétaire Général, Intermédiaire des Recherches Mathematiques, 55 rue de Varenne, Paris 7, France

J. H. BELL, Ph.D. (Wisconsin) Asst. Prof., Michigan State Coll., East Lansing, Mich.

W. D. BERG, Ph.D. (Iowa) Visiting Asst. Prof., Kenyon Coll., Gambier, Ohio

ARTHUR BERNHART, Ph.D. (Michigan) Asst. Prof., Univ. of Oklahoma, Norman, Okla.

R. P. BRADY, M.S. (Chicago) Hunter Coll., New York, N. Y.

W. H. BUNCH, A.M. (Oregon) Head of Dept., High School, Adrian, Ore.

J. J. BURCKHARDT. Prof., Univ. of Zurich, Zurich, Switzerland

G. P. BURNS, M.S. (W. Va. Univ.) Asso. Prof., Marshall Coll., Huntington, W. Va.

GRACE M. CANADA, A.M. (Columbia) Asst. Prof., East Central State Coll., Ada, Okla.

H. N. CARTER, B.S. (Northeastern Okla. St. C.) Asst. Prof., Univ. of Tulsa, Tulsa 4, Okla.

C. L. CLARK, Ph.D. (Virginia) Asso. Prof., Oregon State Coll., Corvallis, Ore.

W. W. S. CLAYTOR, Ph.D. (Pennsylvania) Prof., Hampton Inst., Hampton, Va.

NATHANIEL COBURN, Ph.D. (Mass. Inst. of Tech.) Asst. Prof., Univ. of Michigan, Ann Arbor, Mich.

BROTHER CYPRIAN LUKE (Roney), M.S. (Catholic Univ.) Head of Dept., Sacred Heart Coll., Las Vegas, N. M.

CONSTANCE H. DAVIS (Mrs. Leslie). 238 Audley St., South Orange, N. J.

R. B. DERFLINGER. Student; Asst., Geneva Coll., Beaver Falls, Pa.

ALICE B. DICKINSON, A.M. (Columbia) Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich.

MARY P. DOLCIANA, A.M. (Cornell) Fellow, Cornell Univ., Ithaca, N. Y.

ROY DUBISCH, Ph.D. (Chicago) Asst. Prof., Syracuse Univ., Triple Cities Coll., Endicott, N. Y.

R. W. ENGEL. Hotel Kahler, Rochester, Minn.

D. M. FRIEDLEN. Student, Illinois Inst. of Tech., Chicago, Ill.

C. M. FULTON, Ph.D. (Tech. Hochschule, Munich) Lecturer, Univ. of California at Los Angeles, Los Angeles, Calif.

S. H. GOULD, Ph.D. (Yale) Asso. Prof., Victoria Coll., Toronto, Ont., Canada

E. C. GRAS, A.M. (Harvard) Instr., U. S. Naval Acad., Annapolis, Md.

W. T. GUY, JR., B.S. (A. and M. Coll. of Texas) Instr., Univ. of Texas, Austin, Tex.

W. R. HANSON, M.S. (Chicago) Instr., San Francisco Jr. Coll., San Francisco, Calif.

G. C. HELME, M.S. (Washington Univ.) Instr., Pratt Inst., Brooklyn, N. Y.

R. B. HERRERA, A.M. (U.C.L.A.) Instr., Los Angeles City Coll., Los Angeles, Calif.

V. A. HOERSCH, Ph.D. (Iowa) Asst. Prof., Univ. of Illinois, Urbana, Ill.

CARL HOLTOM, Ph.D. (Chicago) Asst. Prof., Purdue Univ., Lafayette, Ind.

LEROY HOLUBAR, B.S.E.E. (Colorado) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo.

R. E. HORTON, A.B. (U.C.L.A.) Asso., Univ. of California at Los Angeles, Los Angeles, Calif.

H. B. HOYLE, JR., A.M. (North Carolina) Prof., Head of Dept., Queens Coll., Charlotte, N. C.

P. L. HSU, D.Sc. (London) Asso. Prof., Univ. of North Carolina, Chapel Hill, N.C.

ELAINE HUNDERTMARK, A.M. (Illinois) Instr., Univ. of Arkansas, Fayetteville, Ark.

H. V. HUNEKE, A.M. (Oklahoma) Instr., Northwestern State Coll., Alva, Okla.

MRS. FAY H. JOHNSON, A.B. (Howard Payne Coll.) Instr., Howard Payne Coll., Brownwood, Tex.

H. F. S. JONAH, Ph.D. (Purdue) Asso. Prof., Purdue Univ., West Lafayette, Ind.

H. S. KIEVAL, Ph.D. (Cincinnati) Instr., Brooklyn Coll., Brooklyn, N. Y.

W. J. KLIMCZAK, A.M. (Yale) Instr., Yale Univ., New Haven, Conn.

J. T. KRATTIGER, A.M. (Southern Methodist) Instr., Univ. of Oklahoma, Norman, Okla.

R. R. KUEBLER, JR., A.B. (Dickinson Coll.) Instr., Dickinson Coll., Carlisle, Pa.

J. P. LASALLE, Ph.D. (Calif. Inst. of Tech.) Asst. Prof., Univ. of Notre Dame, Notre Dame, Ind.

CHARLES LOEWNER, Ph.D. (Prague) Prof., Syracuse Univ., Syracuse, N. Y.

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R. V. LYNCH, B.S. (Harvard) Instr., Phillips Exeter Acad., Exeter, N. H.

R. A. LYTLE, A.M. (Virginia) Adj. Prof., Univ. of South Carolina, Columbia, S. C.

A. L. MAYERSON, B.S. (Michigan) Grad. Student; Teaching Fellow, Univ. of Michigan, Ann Arbor, Mich.

ELNA B. MCBRIDE (Mrs. J. S.), M.S. (Tennessee) Asst. Prof., Memphis State Coll., Memphis, Tenn.

A. W. MCGAUGHEY, Ph.D. (Cincinnati) Prof., Chm. of Dept., Westminster Coll., New Wilmington, Pa.

W. K. MCNABB, A.M. (Michigan) Instr., The Hockaday School, Dallas 6, Tex.

P. E. MEADOWS, A.B. (Carleton Coll.) Asst. Prof., Washington and Lee Univ., Lexington, Va.

SERGE MINOIS, Agrégation des Sci. Math. (Paris) Asst. Prof., Lyceé Lakanol e Sceaux. 7 rue Candelot, Bourg-la-Reine (Seine), France

A. H. MOORE, B.M.E. (Pratt Inst.) Instr., Pratt Inst., Brooklyn, N. Y.

DOROTHY J. MORROW, M.S. (Washington) Asst. Prof., Statistics, George Washington Univ., Washington, D. C.

D. J. MYATT, B.M.E. (Univ. of Louisville) Asst. Engr., James Clark, Jr. Electric Co., Louisville, Ky.

(MISS) ANDREWA R. NOBLE, Ph.D. (California) Instr., Montana State Univ., Missoula, Mont.

A. M. OSTROWSKI, Ph.D. (Göttingen) Prof., Univ. of Basle, Basle, Switzerland

A. O. QUALLEY, A.M. (Iowa) Instr., Lehigh Univ., Bethlehem, Pa.

MARY K. RAPP, A.M. (Illinois) Instr., Illinois Inst. of Tech., Chicago, Ill.

R. W. RECTOR, A.M. (Stanford) Instr., U. S. Naval Acad., Annapolis, Md.

LOUIS ROSS, A.M. in Educ. (Univ. of Akron) Instr., Univ. of Akron, Akron, Ohio

C. E. RUSCH, A.M. (Wisconsin) Prof., Mission House Coll., R.R. 3, Plymouth, Wis.

R. F. SCHWETYE. Chem. Engr., Mexican Zinc Co., Rosita, Coahuila, Mexico

W. H. SIMONS, M.A. (Univ. of B.C.) Lecturer, Univ. of British Columbia, Vancouver, B. C., Canada

DOROTHY E. SMITH, B.S. in Educ. (N. Ill. St. T.C.) Teacher, Community High School, Erie, Ill.

MALCOLM SMITH. Student, Illinois Inst. of Tech., Chicago, Ill.

L. C. SNIVELY, M.S.E.E. (Colorado) Asst. Prof., Engg. Math., Univ. of Colorado, Boulder, Colo.

O. S. SPEARS, M.S. (Ala. Poly. Inst.) Instr., Univ. of Oklahoma, Norman, Okla.

J. C. STEWART, Ph.D. (Illinois) Asst. Prof., Lawrence Coll., Appleton, Wis.

IRVING SUSSMAN, B.S. (Columbia) Asst. Prof., Univ. of Dayton, Dayton, Ohio

MARGARET O. TAYLOR, M.S. (Pittsburgh) Mathematician, Gulf Research and Development Co., Pittsburgh, Pa.

W. B. TEMPLE, A.M. (Louisiana State) Asst. Prof., A. and M. Coll. of Texas, College Station, Tex. *On leave. Instr., Univ. of Texas, Austin, Tex.*

FEODOR THEILHEIMER, Ph.D. (Berlin) Asst. Prof., Trinity Coll., Hartford, Conn.

LEONARD TORNHEIM, Ph.D. (Chicago) Instr., Univ. of Michigan, Ann Arbor, Mich.

G. R. VICK, A.M. (Sam Houston, St. T.C.) Asst. Prof., Sam Houston State Teachers Coll., Huntsville, Tex.

B. M. WALL, A.M. (Sam Houston, St. T.C.) Asso. Prof., Sam Houston State Teachers Coll., Huntsville, Tex.

MRS. LILLIE C. WALTERS, A.M. (Colorado) Instr., Engg. Math., Univ. of Colorado, Boulder, Colo.

W. R. WASOW, Ph.D. (New York Univ.) Asst. Prof., Swarthmore Coll., Swarthmore, Pa.

G. C. WEBBER, Ph.D. (Chicago) Asso. Prof., Univ. of Delaware, Newark, Del.

HELEN R. WHITE, A.B. (North Carolina) Statistician, Bureau of Census, Dept. of Commerce; Grad. Student, George Washington Univ., Washington, D. C.

A. L. WHITEMAN, Ph.D. (Pennsylvania) Mathematician, Office of Chief of Naval Operations, Washington, D. C.

MRS. BERYL W. WILLIAMS, A.M. (Maine) Instr., Agric. and Tech. Coll., Greensboro, N. C.

FLORENCE A. WIRSCHING, A.B. (N.J.St.T.C., Montclair) Instr., Purdue Univ., West Lafayette, Ind.

L. G. WORTHINGTON, A.M. (N. Tex. State T.C.) Instr., Univ. of Texas, Austin, Tex.

C. B. WRIGHT, Ph.D. (Pittsburgh) Prof., East Texas State Teachers Coll., Commerce, Tex.

H. M. ZERBE, M.S. (Pennsylvania State) Instr., Pennsylvania State Coll., Hazleton Undergraduate Center, Hazleton, Pa.

The Secretary reported the deaths of the following members of the Association:

JOSEPH ALLEN, Associate Professor of mathematics, Retired, College of the City of New York. (March 4, 1946)

DANIEL ARANY, Professor emeritus of mathematics, Royal School for Industry, Budapest.

HARRY BATEMAN, Professor of mathematics, California Institute of Technology. (January 21, 1946)

L. M. BERKELEY, Lawyer, New York, N. Y.

H. F. BLICHFELDT, Professor emeritus of mathematics, Stanford University. (November 16, 1945)

A. A. BLUMBERG, Assistant Professor of mathematics, A. and M. College of Texas. (October 21, 1945)

A. L. CANDY, Professor emeritus of mathematics, University of Nebraska. (July 18, 1945)

F. H. CLUTZ, Professor emeritus of civil engineering, Gettysburg College. (December 30, 1945)

A. R. CONGDON, Professor emeritus of secondary education, University of Nebraska. (November 11, 1945)

A. R. CRATHORNE, Professor emeritus of mathematics, University of Illinois. (March 7, 1946)

FLETCHER DURELL, Head of department emeritus, Lawrenceville School, New Jersey. (March 25, 1946)

C. W. EMMONS, Professor of mathematics, Simpson College. (December 29, 1945)

FEDERIGO ENRIQUES, Professor of mathematics, Retired, University of Rome. (June 14, 1946)

A. M. FREEMAN, Director of mathematical laboratory, Boston Fiduciary and Research Associates. (May 20, 1946)

HANS FRIED, Lecturer in mathematics, Swarthmore College. (December 23, 1945)

C. F. GUMMER, Professor of mathematics, Queen's University, Kingston, Canada. (January 21, 1946)

LAURENCE HADLEY, Professor of mathematics, Purdue University. (March 21, 1946)

DUNHAM JACKSON, Professor emeritus of mathematics, University of Minnesota. (November 6, 1946)

D. E. MARRS, Teacher, Senior High School, Pittsburg, California. (May 28, 1946)

A. S. MCMASTER, Assistant Professor of engineering mathematics, University of Colorado. (May 24, 1946)

T. A. PIERCE, Professor of mathematics, University of Nebraska. (August 18, 1945)

R. G. M. SABEL, Teacher, Bristol (Connecticut) High School. (February 24, 1946)

I. D. STEWART, Associate Professor of mathematics, Whitman College. (July 21, 1946)

G. B. SWEAZEY, Dean, Westminster College, Fulton, Missouri. (August 10, 1946)

ALTHEOD TREMBLAY, Professor of mathematics, Laval University, Quebec, Canada. (March 1, 1946)

E. E. WHITFORD, Professor emeritus of mathematics, College of the City of New York. (May 3, 1946)

The Board voted to accept the invitation from the University of Wisconsin to hold the Summer Meeting of 1948 at Madison, Wisconsin, and the invitation from the University of Colorado to hold the Summer Meeting of 1949 at Boulder, Colorado. Also it was voted to hold the Annual Meeting of 1948 in New York City in conjunction with the meetings of the American Association for the Advancement of Science.

On nominations by the Executive Committee the Board elected C. B. Allendoerfer as Second Vice-President for the two years 1947–48; also G. T. Whyburn as Representative on the National Research Council for the term July 1947 to July 1950, and W. T. Martin as Representative on the Council of the American Association for the Advancement of Science for the two years 1947–48.

The Board approved of the appointment by President MacDuffee of a Nominating Committee for 1947 consisting of W. F. Cheney, Jr., W. M. Whyburn, and L. L. Dines, chairman.

The Board voted to resume the making of exchange arrangements between the MONTHLY and other mathematical periodicals, a practice which had been discontinued by action of the Board on September 12, 1943. Also the Board approved of the plan to place H. M. Gehman in charge of all such exchange arrangements, including the matter of the disposal of all periodicals received by exchange.

The Board appointed C. V. Newsom, R. M. Foster, J. F. Randolph, and W. B. Carver as a committee with power to make a new contract for the printing of the MONTHLY.

On the nomination of the Editor-in-Chief, C. V. Newsom, the Board elected the following Associate Editors of the MONTHLY for the year 1947:

C. B. ALLENDOERFER	H. P. EVANS	W. T. MARTIN
E. F. BECKENBACH	HOWARD EVES	L. F. OLLMANN
L. M. BLUMENTHAL	B. F. FINKEL	R. F. RINEHART
N. B. CONKWRIGHT	B. W. JONES	E. P. STARKE
H. S. M. COXETER	N. H. MCCOY	E. P. VANCE

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday at 9:30 A.M., President MacDuffee presiding.

The following results of the balloting for officers was announced:

L. R. Ford, Illinois Institute of Technology, was elected President for the term 1947–48.

D. H. Lehmer, University of California, and W. T. Martin, Massachusetts Institute of Technology, were elected Governors at Large for the term 1947–49.

The Secretary also announced the election by the Board of C. B. Allendoerfer as Second Vice-President for the term 1947–48.

Upon recommendation by the Board of Governors the following changes in the By-laws of the Association were adopted:

ARTICLE III, Section 8 (a). The words "by this membership in constituencies (hereinafter called 'Regions') established by the Board" shall be replaced by the words "by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist."

ARTICLE III, Section 8 (c). In the first sentence the word "Region" is to be replaced by the word "Section," the word "biennially" by the word "triennially," and the word "two" by the word "three." The second sentence is to be replaced by the sentence, "For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section."

ARTICLE III, Section 8 (g). At the end of the first sentence the phrase "by the Board" shall be replaced by the words "by the President with the approval of the Board." In the second sentence the words "two months" shall be changed to "six months." In the third sentence the word "Board" in both places where it occurs is to be replaced by the words "Nominating Committee."

ARTICLE IV, Section 2. The whole first sentence is to be replaced by the sentence, "The Board shall hold a meeting each year immediately preceding the annual meeting of the Association."

ARTICLE V, Section 2. At the end of the sentence there shall be added the phrase, "except as the Board may provide."

W. B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

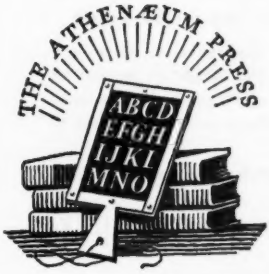
Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA, Cedar Falls, April 18–19, 1947
KANSAS, Wichita, April 19, 1947
KENTUCKY
LOUISIANA-MISSISSIPPI, Hattiesburg, Mississippi, April 25–26, 1947
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Brooklyn, April 19, 1947
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST, Vancouver, British Columbia, April 10–11, 1947
PHILADELPHIA
ROCKY MOUNTAIN
SOUTHEASTERN, Columbia, S. C., April 18–19, 1947
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Rochester, May 10, 1947
WISCONSIN, Madison, May, 1947

Books in Great Demand



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ROSENBACH-WHITMAN-MOSKOVITZ Trigonometries

BALLOU-STEEN Trigonometries

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